

①

$$(x-1)y'' - (x-3)y' - y = 0, \quad y(0) = 3$$
$$y'(0) = 3$$

write  $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow (x-1) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - (x-3) \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} -n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} (n(n-1) + 3n) a_n x^{n-1} + \sum_{n=0}^{\infty} (-n-1) a_n x^n = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} -(m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} ((m+1)m + 3(m+1)) a_{m+1} x^m + \sum_{m=0}^{\infty} (-m-1) a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left( -(m+2)(m+1) a_{m+2} + (m+1)(m+3) a_{m+1} - (m+1) a_m \right) x^m = 0$$

$= 0 \quad \forall m \geq 0$

$$\Rightarrow a_{m+2} = \frac{m+3}{m+2} a_{m+1} - \frac{1}{m+2} a_m$$

ICs:  $a_0 = 3$

$a_1 = 3$

$$\Rightarrow a_2 = \frac{3}{2} a_1 - \frac{1}{2} a_0 = \frac{9}{2} - \frac{3}{2} = 3$$

notice whenever  $a_{m+1} = a_m$

$$a_{m+2} = \frac{m+3}{m+2} a_m - \frac{1}{m+2} a_m = \frac{m+2}{m+2} a_m = a_m$$

$$\Rightarrow a_m = 3, \quad \forall m \geq 0$$

$$\Rightarrow \boxed{y(x) = 3 \sum_{n=0}^{\infty} x^n = \frac{3}{1-x}}$$

$$(2) \quad (x^2+1)y'' + \frac{7}{2}xy' + y = 0$$

write  $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow (x^2+1) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \frac{7}{2}x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n(n-1) + \frac{7}{2}n + 1) a_n x^n = 0$$

$(m=n-2)$    $(m=n)$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=0}^{\infty} (m^2 + \frac{7}{2}m + 1) a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left( (m+2)(m+1)a_{m+2} + (m^2 + \frac{7}{2}m + 1) a_m \right) x^m = 0 \quad \forall x$$

$\underbrace{\hspace{15em}}_{=0} \quad \forall m \geq 0$

$$\Rightarrow a_{m+2} = -\frac{(m+\frac{1}{2})}{m+1} a_m = -\frac{(2m+1)}{2m+2} a_m$$

$$\Rightarrow a_2 = -\frac{1}{2} a_0$$

$$a_4 = \left(-\frac{5}{6}\right) \left(-\frac{1}{2}\right) a_0$$

$$a_6 = \left(-\frac{9}{10}\right) \left(-\frac{5}{6}\right) \left(-\frac{1}{2}\right) a_0$$

⋮

$$a_3 = -\frac{3}{4} a_1$$

$$a_5 = \left(-\frac{7}{8}\right) \left(-\frac{3}{4}\right) a_1$$

$$a_7 = \left(-\frac{11}{12}\right) \left(-\frac{7}{8}\right) \left(-\frac{3}{4}\right) a_1$$

⋮

∴ general solution is :

$$y(x) = a_0 \left( 1 - \frac{1}{2}x^2 + \frac{5 \cdot 1}{6 \cdot 2}x^4 - \frac{9 \cdot 5 \cdot 1}{10 \cdot 6 \cdot 2}x^6 + \dots \right) + a_1 \left( x - \frac{3}{4}x^3 + \frac{7 \cdot 3}{8 \cdot 4}x^5 - \frac{11 \cdot 7 \cdot 3}{12 \cdot 8 \cdot 4}x^7 + \dots \right)$$

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$$y'' + x^2 y' + y = 0$$

write  $y = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

( $m=n-2$ )

( $m=n+1$ )

( $m=n$ )

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=1}^{\infty} (m-1)a_{m-1} x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

(take out  $n=0$  term)

$$(2 \cdot 1 \cdot a_2 + a_0) + \sum_{m=1}^{\infty} ((m+2)(m+1)a_{m+2} + (m-1)a_{m-1} + a_m) x^m = 0$$

↓

= 0

$$a_2 = -\frac{1}{2} a_0$$

$$a_{m+2} = \frac{-1}{(m+2)(m+1)} a_m - \frac{(m-1)}{(m+2)(m+1)} a_{m-1}, \quad m \geq 1$$

$$\Rightarrow a_3 = -\frac{1}{6} a_1$$

$$\Rightarrow a_4 = \frac{-1}{12} a_2 - \frac{1}{12} a_1$$

$$a_4 = \frac{1}{24} a_0 - \frac{1}{12} a_1$$

$$\& a_5 = \frac{-1}{20} a_3 - \frac{1}{10} a_2$$

$$a_5 = \frac{1}{20} a_1 + \frac{1}{20} a_0$$

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a)

~~$p(x) = \frac{3}{x^2}$~~   $\Rightarrow x=0$  singular point

$$\lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} \frac{3}{x} \text{ DNE} \Rightarrow \boxed{x=0 \text{ irregular singular point}}$$

b)  $p(x) = \frac{x+1}{(x^2-1)^2} = \frac{x+1}{(x+1)^2(x-1)^2} = \frac{1}{(x+1)(x-1)^2}$

$\Rightarrow x = -1, 1$  singular points

$$\lim_{x \rightarrow -1} (x+1)p(x) = \lim_{x \rightarrow -1} \frac{1}{(x-1)^2} = \frac{1}{4}, \text{ finite}$$

$$\lim_{x \rightarrow -1} (x+1)^2 p(x) = \lim_{x \rightarrow -1} (x+1)^2 \frac{x+2}{(x^2-1)^2}$$

$$= \lim_{x \rightarrow -1} \frac{x+2}{(x-1)^2} = \frac{1}{4} \text{ finite}$$

$\Rightarrow \boxed{x=-1 \text{ regular singular point}}$

$$\lim_{x \rightarrow 1} (x-1)p(x) = \lim_{x \rightarrow 1} \frac{1}{(x+1)(x-1)} \text{ DNE}$$

$\Rightarrow \boxed{x=1 \text{ irregular singular point}}$

c)  $g(x) = \frac{2}{\sin^2(x)} \Rightarrow x$  singular whenever

$$\sin(x) = 0$$

$$\Rightarrow x = n\pi, n \in \mathbb{Z} \text{ singular}$$

$$\lim_{x \rightarrow n\pi} (x - n\pi)^2 g(x) = \lim_{x \rightarrow n\pi} \frac{2(x - n\pi)^2}{\sin^2(x)}$$

$$= \lim_{x \rightarrow n\pi} \frac{4(x - n\pi)}{2 \sin(2x) \cos(x)}$$

L'Hopital's rule

$$= \lim_{x \rightarrow n\pi} \frac{4(x - n\pi)}{\sin(2x)}$$

L'Hopital's rule

$$= \lim_{x \rightarrow n\pi} \frac{4}{2 \cos(2x)}$$

$$= \frac{4}{2 \cos(2n\pi)}$$

$$= 2, \text{ finite, } \forall n$$

$$\lim_{x \rightarrow n\pi} (x - n\pi) p(x) = \lim_{x \rightarrow n\pi} \frac{(x - n\pi) \sin\left(\frac{x}{2}\right)}{\sin^2(x)}$$

L'Hopital's rule

$$= \lim_{x \rightarrow n\pi} \frac{\sin\left(\frac{x}{2}\right) + (x - n\pi) \frac{1}{2} \cos\left(\frac{x}{2}\right)}{\sin(2x)}$$

1) if  $n$  is odd  $\sin\left(\frac{x}{2}\right) \rightarrow \sin\left(\frac{n\pi}{2}\right) = 1$  or  $-1$

$$\Rightarrow \text{limit ONE (const/0 case)}$$

2) if  $n$  is even (0/0 case)

$$= \lim_{x \rightarrow n\pi} \frac{\cos\left(\frac{x}{2}\right) - (x - n\pi) \frac{1}{4} \sin\left(\frac{x}{2}\right)}{2 \cos(2x)}$$

by L'Hopital's rule

$$= \frac{\cos\left(\frac{n\pi}{2}\right)}{2 \cos(2n\pi)} \text{ is finite}$$

The functions have Taylor series

thus	$x = n\pi$	regular singular point	if $n$ is even
		irregular singular point	if $n$ is odd

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a)  $p(x) = \frac{1}{x^2-3}$  has singularities at  $x = \pm\sqrt{3}$

$\Rightarrow$  radius of convergence of Taylor series of  $p(x)$  centred at  $x_0 = 0$  is  $|0 - \sqrt{3}| = \sqrt{3}$

$q(x) = \frac{\tan(x)}{x^2-3}$  has singularities at  $x = \pm\sqrt{3}, (n+\frac{1}{2})\pi, \forall n \in \mathbb{Z}$

$\Rightarrow$  radius of convergence of Taylor series of  $q(x)$  centred at  $x_0 = 0$  is  $|0 - \frac{\pi}{2}| = \frac{\pi}{2}$  (since  $\frac{\pi}{2}$  (or  $-\frac{\pi}{2}$ ) is the closest singularity to  $x_0 = 0$ )

Thus radius of convergence of

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \geq \min(\sqrt{3}, \frac{\pi}{2}) = \boxed{\frac{\pi}{2}}$$

b)  $p(x) = \frac{x^3}{4x^2+9}$  has singularities at

$$x = \pm \sqrt{\frac{-9}{4}} = \pm \frac{3i}{2}$$

and same for  $q(x)$

$\Rightarrow$  radius of convergence of  $p(x)$  &  $q(x)$  centred at  $x_0 = 2$  is

$$|2 - \frac{3i}{2}| = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

$\Rightarrow$  radius of convergence of series ~~representation~~ representation of  $y(x)$  centred at  $x_0 = 2$  is at least  $\boxed{\frac{5}{2}}$