## 1 Math 406, Project 1 Due Nov 16 th: Mold filling problems

## 2 Problem description

We consider a class of boundary value problems (BVP) for $p(r, t)$ of the form

$$
\begin{align*}
\qquad \Delta p= & D \frac{1}{r}\left(r p_{r}\right)_{r}=g(r, t) \text { where } r_{0}<r<R(t)  \tag{1}\\
\text { Left BC: } & \text { Specified pressure: } p\left(r_{0}, t\right)=p_{0}  \tag{2}\\
\text { Right BC }: & p(R(t), t)=0  \tag{3}\\
\text { Initial Condition }: & p(r, 0)=0 \text { at } t=0 \tag{4}
\end{align*}
$$

Here $\Delta p$ is the Laplacian, which reduces to an ordinary differential operator since the problem is assumed to be radially symmetric. This boundary value problem represents the pressure distribution within a circular fluid-filled zone occupying the cylindrical region $r_{0}<r<R(t),-\frac{w_{0}}{2}<z<\frac{w_{0}}{2}$ between two parallel plates a distance $w_{0}$ apart and $r_{0}$ is the radius of the tube through which the fluid is pumped. The flow velocity, according to Poiseuille's law, is given by

$$
\begin{equation*}
v=-\frac{w_{0}^{2}}{\mu^{\prime}} \frac{d p}{d r} \tag{5}
\end{equation*}
$$

Here $\mu^{\prime}=12 \mu$ is the scaled fluid viscosity and the velocity is obtained by integrating the parallel plate solution to the Navier Stokes equations in the $z$ directions across the gap between the plates. Associated with the Poiseuille velocity is the fluid flux within the parallel disks, which is given by

$$
q=w_{0} v=-\frac{w_{0}^{3}}{\mu^{\prime}} \frac{d p}{d r}=-D \frac{d p}{d r}, \text { where } D=\frac{w_{0}^{3}}{\mu^{\prime}}
$$

The ODE (1) expresses the conservation of mass in which the flux gradient is balanced by the sources or sinks $g(r, t)$ distributed within the expanding domain $r_{0}<r<R(t)$. Though the BVP is relatively simple to solve, the fact that the extent of the domain is unknown complicates the problem considerably. This type of problem is known as a "free boundary problem" or "moving boundary problem". At the moving front Poiseuille's law provides the so-called Stefan condition for the front velocity:

$$
\begin{equation*}
\dot{R}(t)=q(R(t)) / w_{0} \tag{6}
\end{equation*}
$$

## 3 Simple solutions without distributed sources or sinks

Assuming no distributed source/sink term (i.e. $g(r, t)=0$ ) determine the pressure distribution $p(r, t)$, the velocity $\dot{R}(t)$ of the moving front, and an expression for $R(t)$. Plot the exact solutions $R(t)$ for the parameters: $w_{0}=\mu^{\prime}=p_{0}=1$ and $r_{0}=0.1$ over the time interval $0<t<200$. You will need to use Newton's method to achieve this. Now plot $p(r, t=200)$.

## 4 Mold filling with fluid loss

Assume that fluid is being lost through the parallel plates via a diffusion process that leads to a sink term of the form

$$
g(r, t)=\frac{C^{\prime} H\left(t-t_{0}(r)\right)}{\sqrt{t-t_{0}(r)}}
$$

where $H(t)$ is the Heaviside function, $C^{\prime}$ is a given constant, and $t_{0}(r)$ is the time at which the fluid font arrives at the ring of radius $r$. Thus $t_{0}(r)=R^{-1}(r)$ the inverse function of $R(t)$.

1. Use the Green's function corresponding to the ODE (1) and the boundary conditions (2-3)to determine an expression for $p(r, t)$ in terms of $R(t)$ and $g(r, t)$.
2. Use this expression to determine an expression for $p_{r}(R)$. Now use the Stefan condition (6) to derive an expression for the front velocity $\dot{R}(t)$. In the integral that results use the transformation of variables $\rho=R(\tau), d \rho=\dot{R}(\tau) d \tau$ to arrive at an Abel integral equation for $\phi(R, \dot{R})=R \log \left(R / r_{0}\right) \dot{R}$ of the form:

$$
\begin{equation*}
\phi(R, \dot{R})=A+B \int_{0}^{t} \frac{\phi(R(\tau), \dot{R}(\tau))}{\sqrt{t-\tau}} d \tau \tag{7}
\end{equation*}
$$

3. Since this integral equation is in the form of a Laplace Transform convolution take the Laplace transform of (7) to determine Laplace transform of $\phi(t)$. The Laplace transform $\mathcal{L}\left(\frac{1}{t^{1 / 2}}\right)=\left(\frac{\pi}{s}\right)^{1 / 2}$ may prove useful.
4. Now invert the Laplace transform of $\phi$ to determine an expression for $\dot{R}(t)$ and thence an expression for $R(t)$. The inverse Laplace Transform $\mathcal{L}^{-1}\left(\frac{1}{s+\alpha s^{1 / 2}}\right)=e^{\alpha^{2} t} \operatorname{erf} c\left(\alpha t^{1 / 2}\right)$ and the integral

$$
\int_{0}^{t} e^{\alpha^{2} \tau} \operatorname{erf} c\left(\alpha \tau^{1 / 2}\right) d \tau=\frac{1}{\alpha^{2}}\left(e^{\alpha^{2} t} \operatorname{erf} c\left(\alpha t^{1 / 2}\right)-1\right)+\frac{2 t^{1 / 2}}{\alpha \pi^{1 / 2}}
$$

may prove useful. Now obtain an expression for $p(r, t)$.
5. Assuming $w_{0}=\mu^{\prime}=C^{\prime}=p_{0}=1$, and $r_{0}=0.1$ plot $R(t)$ for $0<t<200$. Now use quadgk to determine $p(r, t)$ and plot $p(r, t)$ at $t=200$.

## 5 Asymptotics and scaling

### 5.1 Scaling

By dimensional analysis and scaling it is frequently possible to derive the fundamental power law relationships between the different variables in the model.

For the boundary value problem is of the form:

$$
\begin{equation*}
D \frac{1}{r}\left(r p_{r}\right)_{r}=\frac{C^{\prime} H\left(t-t_{0}(r)\right)}{\sqrt{t-t_{0}(r)}} \tag{8}
\end{equation*}
$$

Introduce characteristic length, time, and pressure scales $R_{*}, t_{*}$ and $p_{*}=p_{0}$ and dimensionless variables

$$
x=R_{*} \rho, t=t_{*} \tau, p(r, t)=p_{0} \Pi(\rho, \tau),
$$

to reduce (8) to the form

$$
\mathcal{G}_{\mu} \frac{1}{\rho}\left(\rho \Pi_{\rho}\right)_{\rho}=\frac{H\left(\tau-\tau_{0}\right)}{\sqrt{\tau-\tau_{0}}}
$$

By requiring $\mathcal{G}_{\mu}=1$ determine $\gamma$ in the power law relationship $R_{*}{ }^{\sim} t_{*}^{\gamma}$. Include this asymptotic estimate in your plot of the exact solution.

