MATH 406, HWK 6, Due 23 November 2018

1. Consider the following Dirichlet boundary value problem for Laplace's equation on a quarter circle.

$$\begin{aligned} \Delta u &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,\ 0 < r < a,\ 0 < \theta < \pi/2 \qquad (1) \\ u(r,0) &= 0, \quad u(r,\pi/2) = 0,\ u(a,\theta) = f(\theta) = \sin(8\theta) \end{aligned}$$

   (a) Show that the exact solution to (1) is:

$$u(r,\theta) = \left(\frac{r}{a}\right)^8 \sin(8\theta)$$

   (b) Alter the program `poisson.m` (a copy of `poisson.m, V.m,` and `VN.m` can be found on the course web site) to be able to solve this boundary value problem using the direct boundary element method (DBEM). Assuming that $a = 10$ compare the exact and numerical solutions along the line of benchmarks

$$\theta = \pi/16 \text{ and } r = 0.1 : 0.1 : a - 0.1$$

   (c) Implement the indirect boundary element method (IBEM) using the single layer potential to solve Laplace's equation:

$$u(x) = \int_{\partial\Omega} F(x')G(x', x)\, ds(x') \qquad (2)$$

   Again assuming $a = 10$ use the IBEM to solve (1). Compare the exact and IBEM solutions along the same line of benchmarks as in (b).

2. To model wave propagation in microstrip antennae it is necessary to determine the Green's function for multilayered dielectric structures. In this example, which serves as a prototype for multilayer Green's functions, we consider the Green's function for two bonded half-spaces each having different permittivities. Use the two dimensional Fourier Transform to determine the Green's function for a point source a distance h below the plane $z = 0$ between the two bonded half-spaces. The governing equatons are:

$$\nabla \cdot (\varepsilon(z)\nabla G(x, y, z) = \delta(x)\delta(y)\delta(z + h), \quad -\infty < x, y, z < \infty$$

   where

$$\varepsilon(z) = \begin{cases} \varepsilon_2 \text{ for } z < 0 \\ \varepsilon_1 \text{ for } z > 0 \end{cases}$$

   Execute the following procedure:

(a) Use the two dimensional Fourier Transform

$$\mathcal{F}(u) = \widehat{u}(m,n,z) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{i(mx+ny)} u(x,y,z)dxdy$$

to reduce the above boundary value problem for G to an ODE in z for $\widehat{G}_j$ in each layer of the form: $\varepsilon_j \left(\widehat{G}_{j,zz} - k^2\widehat{G}_j\right) = \delta(z+h)$ where $k = \sqrt{m^2+n^2}$ and $\widehat{G}_1$ is the Greens function for the upper half-space and $\widehat{G}_2$ is the Green's function for the lower half-space. $\widehat{G}_1$ involves only a decaying exponential e^{-kz}, while $\widehat{G}_2$ involves both exponentials e^{-kz} and e^{kz} in the strip $-h < z < 0$, and the decaying exponential e^{kz} in the half-space $-\infty < z < -h$.

(b) Now use the following stitching process, similar to the one introduced for Green's functions for ODEs, to determine the four coefficients of the exponential solutions that define $\widehat{G}_j(k,z)$. There should be continuity in values and fluxes for $\widehat{G}_1$ and $\widehat{G}_2$ across the plane $z = 0$. Likewise, the values of $\widehat{G}_2$ should be continuous across the plane $z = -h$, and the fluxes in $\widehat{G}_2$ across the plane $z = -h$ should jump to match the strength of the delta function source.

(c) Now use the Fourier inversion formula

$$G_j(x,y,z) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{i(mx+ny)}\widehat{G}(k,y)dmdn$$

to determine $G_j(x,y,z)$.

(d) Can you interpret the solution in terms of sources and images?