1. Consider the following dirichlet boundary value problem for Laplace’s equation on a quarter circle.

\[ \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \ 0 < r < a, \ 0 < \theta < \pi/2 \]

\[ u(r,0) = 0, \ u(r,\pi/2) = 0, \ u(a,\theta) = f(\theta) = \sin(8\theta) \]

a. Show that the exact solution to (i) is:

\[ u(r,\theta) = \left( \frac{r}{a} \right)^8 \sin(8\theta) \]

b. Alter the program `poisson.m` (a copy of `poisson.m`, `V.m`, and `VN.m` can be found on the course web site) to be able to solve this boundary value problem using the direct boundary element method (DBEM). Assuming that \( a = 10 \) compare the exact and numerical solutions along the line of benchmarks

\[ \theta = \pi/16 \text{ and } r = 0.1 : 0.1 : a - 0.1 \]

c. Implement the indirect boundary element method (IBEM) using the single layer potential to solve Laplace’s equation:

\[ u(x) = \int_{\partial \Omega} F(x')G(x',x)ds(x') \]

Again assuming \( a = 10 \) use the IBEM to solve (i). Compare the exact and IBEM solutions along the same line of benchmarks as in (b).

Solution: (b)
Direct BEM solution to Laplace Equation using $N = 14$, $u = 0$ along $\theta = \pi/16, 0 < r < 10$, $N = 14$
Direct BEM solution to Laplace Equation using $N = 14$ Error along $\theta = \pi/16, 0 < r < 10, N = 14$
2) Use the one dimensional Fourier Transform

a. 
\[ \mathcal{F}(u) = \hat{u}(k,y) = \int_{-\infty}^{\infty} e^{ikx}u(x,y)dx \]

to reduce the above boundary value problem for \( G \) to an ODE for \( \hat{G} \).

b. Now use the stitching process introduced for Green’s functions for ODEs to determine the solution \( \hat{G}(k,y) \) to this ODE and boundary conditions - assuming that \( \hat{G}(k,y) \to 0 \) as \( y \to \infty \).

c. Now use the Fourier inversion formula \( G(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\hat{G}(k,y)dk \) to determine \( G(x,y) \).

d. Use the free space Green’s function in 2D to interpret your result, , i.e. recall \( G = \frac{1}{2\pi} \ln|\vec{x}' - \vec{x}| + h(\vec{x}') \) where \( \Delta h = 0 \) and \( h(\vec{\xi},0) = -\frac{1}{2\pi} \ln[(\vec{\xi} - x)^2 + y^2]^{1/2} \).
Q2: There is no loss of generality in assuming the source is at \((0, 0)\).

\[ \Delta G(x, y) = S(x) S(y) \delta(x) \delta(y) \]

\[ G(x, 0) = 0. \]

\( G(x, y) = 0. \)

a) Take the FT of \( G(x, y) \):

\[ \hat{G}(k_x, k_y) + \hat{C} G_{yy} = -S(k_x, k_y) \]

\[ \hat{G}_{yy} - k^2 \hat{G} = S(k_x, k_y) \]

The homogeneous EQ has a solution \( \hat{G} = A e^{k_x} + B e^{-k_x} \) \( k = 1 \sqrt{} \)

The Greens function we seek has the form

\[ \hat{G}(k_x, k_y) = \begin{cases} A e^{k_x} + B e^{-k_x} & y < h \\ C e^{-k_y} & y > h \end{cases} \]

b) Continuity:

\[ \hat{G}(k_x, k_y) = \begin{cases} A e^{k_x} + B e^{-k_x} & y < h \\ C e^{-k_y} & y > h \end{cases} \]

Jump

\[ \hat{G}_{y} \bigg|_{y=0} = A e^{k_x} - B e^{-k_x} = 1 \]

Boundary condition:

\[ \hat{G}(k_x, 0) = A + B = 0 \quad B = -A \]

1. \( \hat{G}(k_x, y) = C e^{-k_y} 
\]
2. \( C e^{-k_y} - A e^{k_x} + A e^{-k_x} = 0 \)
3. \( k e^{-k_y} - A e^{k_x} + A e^{-k_x} = 0 \)
4. \( -k e^{-k_y} - A e^{k_x} - A e^{-k_x} = 1 \)
5. \( -2k e^{-k_y} = 1 \quad A = -\frac{e^{-k_y}}{2k} = -B \)
6. \( C = A (e^{kh} - 1) = -\frac{1}{2k} (e^{kh} - e^{-kh}) \)

\[ \hat{G}(k_x, 0) = \begin{cases} e^{-k_y} - e^{k_y} & y < h \\ e^{k_y} & y > h \end{cases} \]

\[ \hat{G}(k_x, y) = -\frac{1}{2k} \int \left[ e^{-k_y} - e^{k_y} \right] dk \]

\[ = -\frac{1}{2} \int_0^{\infty} e^{-k_y} dk - \frac{1}{2} \int_0^{\infty} e^{k_y} dk \cos(ky) dk \]

Taking the Laplace transform we obtain

\[ \hat{G}(x, y) = \frac{1}{2\pi} \int \frac{y^2 + r^2}{r^2 + (y - h)^2} \frac{1}{2} - \frac{y^2 + r^2}{r^2 + (y + h)^2} \frac{1}{2} \]

\[ = \frac{1}{2\pi} \left[ \mathcal{H}(r^2 + (y-h)^2)^{1/2} - \mathcal{H}(r^2 + (y+h)^2)^{1/2} \right] \]

Source point, image point
e. Use the DBEM developed in problem 1 (b) to approximate the Green’s Function for a half-space

\[\Delta G(\tilde{x}', \tilde{x}) = \delta(\tilde{x}' - \tilde{x}) \quad \tilde{x}' = (\xi, \eta) \quad \eta > 0\]

\[G(0, \tilde{x}) = 0 \quad \tilde{x}' = (\xi, 0)\]

Discretize the boundary \( \eta = 0 \), \( \xi \in (-10, 10) \) and impose the condition (*) above to find \( h(\xi, \eta) \). Evaluate the numerical and exact Green’s functions along a line of benchmarks \( x = -10 : 1 : 10, y = 1 \) if the delta function \( \delta(\xi - 5, \eta - 2) \) has a source point at \((5, 2)\) and plot the result.

Solution (d)