1. Consider determining the eigenvalues for Laplace’s equation:

\[-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u\]  

(1)

on the region \(\Omega\) subject to Dirichlet boundary condition:

\[u|_{\partial\Omega} = 0\]  

(2)

Starting with the weighted residual statement of this boundary value problem determine the appropriate weak statement of the problem. Consider a triangular tesselation of the domain \(\Omega\) into triangles \(T_k\) such that \(\Omega = \bigcup T_k\). Now use the piecewise linear basis functions \(\psi_k(x, y)\) defined on the triangles \(T_k\), which we developed in class, to arrive at a Galerkin approximation to this eigenvalue problem.

Now adapt the code supplied in \texttt{FEM_packD.zip} by altering the function \texttt{fempoiD.m} so that it calculates the eigenvalues of the Dirichlet problem on a unit circle. Determine the first 6 nonzero eigenvalues for the unit circle \(\Omega = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq 2\pi\}\) using \(n=32\) elements and the meshing parameter \(h_{\text{max}}=2\pi/n\). Use \texttt{tplot.m} to plot the eigenfunctions associated with \(\lambda_{0,1}, \lambda_{1,1}, \lambda_{2,1}\) and \(\lambda_{0,2}\). Provide your code for the FEM formulation of the eigenvalue problem.

Note: Recall that the eigenvalues and eigenfunctions for the Dirichlet problem for a circle of radius \(a\) are given by:

\[\lambda_{m,n} = \left(\frac{j_{m,n}}{a}\right)^2, \quad m = 0, 1, 2, \ldots, n = 1, 2, \ldots\]

\[\phi_{m,n}(r, \theta) = J_m(\frac{j_{m,n}}{a}) \begin{cases} 1 & \text{for } m = 0 \\ \cos m\theta & \\ \sin m\theta & \end{cases}\]

where \(j_{m,n}\) is the \(n\)-th positive zero of the \(m\)–th Bessel \(J_m\). Thus

\[
\lambda_{m,n} = \begin{bmatrix}
\lambda_{0,1} = (2.4048/1)^2 = 5.7831 \\
\lambda_{1,1} = (j_{1,1}/1)^2 = (3.8317)^2 = 14.682 \\
\lambda_{2,1} = (j_{2,1}/1)^2 = (5.1356)^2 = 26.374 \\
\lambda_{0,2} = (j_{0,2}/1)^2 = (5.5201)^2 = 30.472 
\end{bmatrix}
\]

Using \(n=32\) nodes on the boundary in your FEM code for the Dirichlet problem complete the following table of Eigenvalues

<table>
<thead>
<tr>
<th>(\lambda_{m,n})</th>
<th>Exact</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_{0,1})</td>
<td>5.7831</td>
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