MATH 406, HWK 5, Due 14 November 2018

1. Consider determining the eigenvalues for Laplace's equation:

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u \tag{1}$$

on the region Ω subject to Dirichlet boundary condition:

$$u]_{\partial\Omega} = 0 \tag{2}$$

Starting with the weighted residual statement of this boundary value problem determine the appropriate weak statement of the problem. Consider a triangular tesselation of the domain Ω into triangles T^k such that $\Omega = \cup T^k$. Now use the piecewise linear basis functions $\psi_\alpha^k(x, y)$ defined on the triangles T^k, which we developed in class, to arrive at a Galerkin approximation to this eigenvalue problem.

Now adapt the code supplied in `FEM_PackD.zip` by altering the function `fempoiD.m` so that it calculates the eigenvalues of the Dirichlet problem on a unit circle. Determine the first 6 nonzero eigenvalues for the unit circle $\Omega = \{(r, \theta) : r \leq 1, 0 \leq \theta \leq 2\pi\}$ using `n=32` elements and the meshing parameter `hmax=2`π`/n`. Use `tplot.m` to plot the eigenfunctions associated with $\lambda_{0,1}, \lambda_{1,1}, \lambda_{2,1}$ and $\lambda_{0,2}$. Provide your code for the FEM formulation of the eigenvalue problem.

Note: Recall that the eigenvalues and eigenfunctions for the Dirichlet problem for a circle of radius a are given by:

$$\begin{aligned} \lambda_{m,n} &= (j_{m,n}/a)^2, \ m = 0, 1, 2, \ldots, n = 1, 2, \ldots \\ \phi_{m,n}(r, \theta) &= J_m(j_{m,n}/a) \begin{cases} 1 \text{ for } m = 0 \\ \cos m\theta \\ \sin m\theta \end{cases} \end{aligned}$$

where $j_{m,n}$ is the n-th positive zero of the $m-$th Bessel J_m. Thus

$$\lambda_{m,n} = \begin{bmatrix} \lambda_{0,1} = (2.4048/1)^2 = 5.783\,1 \\ \lambda_{1,1} = (j_{1,1}/1)^2 = (3.8317)^2 = 14.682 \\ \lambda_{2,1} = (j_{2,1}/1)^2 = (5.1356)^2 = 26.374 \\ \lambda_{0,2} = (j_{0,2}/1)^2 = (5.5201)^2 = 30.472 \end{bmatrix}$$

Using `n=32` nodes on the boundary in your FEM code for the Dirichlet problem complete the following table of Eigenvalues

$\lambda_{m,n} =$

Exact	FEM
$\lambda_{0,1} = 5.7831$	
$\lambda_{1,1} = 14.682$	
$\lambda_{2,1} = 26.374$	
$\lambda_{0,2} = 30.472$	