1. Consider the following boundary value problem

\[ \mathcal{L}u = u'' + k^2 u(x) = f(x), \quad u(0) = \alpha, \quad u'(1) = \beta \quad (1) \]

(a) Use integration by parts to obtain the weak statement of the BVP.

(b) If \( f \) is sufficiently differentiable show that the strong and weak formulations of the BVP are equivalent.

(c) Write down the Galerkin formulation of this BVP.

(d) Use piecewise linear basis functions and the weak formulation to obtain a Finite Element discretization of the BVP.

(e) Solve the BVP with \( f(x) = x^3 \), \( k = 10 \), \( \alpha = 0 \), and \( \beta = 1 \) and compare with the exact solution.

(f) In the case \( f(x) = 0 \), \( \alpha = 0 \), and \( \beta = 0 \) we have an eigenvalue problem. By considering the minimization of the appropriate Rayleigh quotient or the appropriate weak formulation, use Finite Elements to discretize the problem and to reduce it to a corresponding generalized matrix eigenvalue problem of the form \( Ax = \lambda B x \). Use the MATLAB function \([V,D] = \text{EIG}(A,B)\) to determine the eigenvalues and corresponding eigenvectors for \( N = 10, 20, 30 \) and compare with the results of the FEM and FD solutions by providing the following plots: \( k_j \) vs mode number \( j \), and the first three eigenfunctions.

(g) Formulate (1) as a minimization problem for an appropriate functional. By taking variations of this functional show that the necessary conditions are the same as the weak statement obtained in (a).

2. Consider the strong form of the boundary value problem for a thin Bernoulli-Euler beam that is fixed at one end and free at the other and subjected to an external load \( f(x) \in C^0(-1,1) \). Find a function \( w \in C^4[-1,1] \) which satisfies the following differential equation and boundary conditions:

\[
\begin{align*}
  w_{xxxx} &= f(x) \quad \text{for} \quad -1 < x < 1 \\
  w(-1) &= 0 \quad \text{and} \quad w_x(-1) = 0 \\
  w_{xx}(1) &= M \quad \text{and} \quad w_{xxx}(1) = V
\end{align*}
\]

(a) Derive a weak statement for the boundary value problem detailing the spaces to which \( w(x) \) and the test functions \( v(x) \) should belong.

(b) Assuming that \( w \in C^4[-1,1] \) and that \( w \) is a solution to the weak form derived in (a), show that \( w \) is also a solution of the strong form of the problem.