## MATH 406, HWK 4, Due 7 November 2018

1. Consider the following boundary value problem

$$
\begin{equation*}
\mathcal{L} u=u^{\prime \prime}+k^{2} u(x)=f(x), \quad u(0)=\alpha, \quad u^{\prime}(1)=\beta \tag{1}
\end{equation*}
$$

(a) Use integration by parts to obtain the weak statement of the BVP.
(b) If $f$ is sufficiently differentiable show that the strong and weak formulations of the BVP are equivalent.
(c) Write down the Galerkin formulation of this BVP.
(d) Use piecewise linear basis functions and the weak formulation to obtain a Finite Element discretization of the BVP.
(e) Solve the BVP with $f(x)=x^{3}, k=10, \alpha=0$, and $\beta=1$ and $N=10,20,30$ and compare with the exact solution.
(f) In the case $f(x)=0, \alpha=0$, and $\beta=0$ we have an eigenvalue problem. By considering the minimization of the appropriate Rayleigh quotient or the appropriate weak formulation, use Finite Elements to discretize the problem and to reduce it to a corresponding generalized matrix eigenvalue problem of the form $A x=\lambda B x$. Use the MATLAB function $[\mathrm{V}, \mathrm{D}]=\operatorname{EIG}(\mathrm{A}, \mathrm{B})$ to determine the eigenvalues and corresponding eigenvectors for $N=10$. Discretize the same problem using finite differences, determine an explicit expression for the approximate eigenvalues of the finite difference equations, and determine the order of the error. Compare the results of the FEM and FD solutions by providing the following plots: $k_{j}$ vs mode number $j$, and the first three eigenfunctions.
(g) Formulate (1) as a minimization problem for an appropriate functional. By taking variations of this functional show that the necessary conditions are the same as the weak statement obtained in (a).
2. Consider the boundary value problem that determines $p(r, t)$ for the mold filling problem described in Project 1

$$
\begin{align*}
\frac{1}{r}\left(r p_{r}\right)_{r}= & 0 \text { where } r_{0}<r<R(t)  \tag{2}\\
\text { Left BC: } & \text { Specified pressure: } p\left(r_{0}, t\right)=p_{0}  \tag{3}\\
\text { Right BC }: & p(R(t), t)=0 \tag{4}
\end{align*}
$$

The flow velocity, according to Poiseuille's law, is given by

$$
\begin{equation*}
v=-\frac{w_{0}^{2}}{\mu^{\prime}} \frac{d p}{d r} \tag{5}
\end{equation*}
$$

Here $\mu^{\prime}=12 \mu$ is the scaled fluid viscosity and $w_{0}$ is the distance between the plates of the mold. Associated with the Poiseuille velocity is the fluid flux within the parallel disks, which is given by

$$
q=w_{0} v=-\frac{w_{0}^{3}}{\mu^{\prime}} \frac{d p}{d r}=-D \frac{d p}{d r}, \text { where } D=\frac{w_{0}^{3}}{\mu^{\prime}}
$$

(a) Assuming that $R(t)$ is known use integration by parts to obtain the weak statement of the BVP (2)-(4).
(b) Using piecewise linear finite elements write down the Galerkin formulation for the problem.
(c) For $R=10, r_{0}=0.1$ and $p_{0}=1$ determine the solution to this problem using the Galerkin approximation determined in (b). Compare your results to the exact solution using $N=10$ elements and plot the error.
(d) This type of problem is known as a "free boundary problem" or "moving boundary problem". At the moving front Poiseuille's law provides the so-called Stefan condition for the front velocity:

$$
\begin{equation*}
\left.\dot{R}(t)=q / w_{0}=v=-\frac{w_{0}^{2}}{\mu^{\prime}} \frac{d p}{d r}\right]_{r=R} \tag{6}
\end{equation*}
$$

Using the exact solution along with the Stefan condition evaluated as $r \rightarrow R$, determine a simple ODE for $R$ and an expression defining $R$ implicitly in terms of $t$ assuming $R(0)=r_{0}$. Now adapt your code developed in (c) to solve the free boundary problem by marching forward Ntime steps in time and iterating Nitfront times on the location of the front:
for $k=1$ : Ntime

- $\quad t=t+\Delta t$
$v_{k+1}=v_{k}$
$R_{k+1}^{o}=R_{k}$
for itf $=1$ : Nitfront
$R_{k+1}=R_{k}+\Delta t v_{k+1}$
if $\left|R_{k+1}^{o}-R_{k+1}\right|<t o l \cdot R_{k+1}$ break
Set up FEM mesh for $\left[r_{0}, R_{k+1}\right]$ and solve for $p^{k+1}=$ $\left[p_{1}=p_{0}, p_{2}, \ldots, p_{N+1}=0\right]$
$v_{k+1}=-\frac{w_{0}^{2}}{\mu^{\prime}}\left(\frac{p_{N+1}-p_{N}}{\Delta r_{N}}\right)$
$R_{k+1}^{o}=R_{k+1}$
end (front iteration loop)
end (time step loop)
Where $\Delta r_{N}=r_{N+1}-r_{N}=R-r_{N}$. Assume $w_{0}=1, \mu^{\prime}=1$, and $p_{0}=1$ and plot $t$ vs $R(t)$ for $t \in[0,5]$. Since $R$ is only defined implicitly as a function of $t$, rather than solving for $R$ using Newton's method, evaluate $t$ as a function of $R$ and use the pchip function as follows to define $R(t)$ :

$$
\mathrm{R}=@(\mathrm{~s}) \mathrm{ppval}(\mathrm{pchip}(\mathrm{ts}, \mathrm{rs}), \mathrm{s}) ;
$$

Also plot the exact and FEM solution $p(r, 5)$ vs $r$ for $r \in\left[r_{0}, R(5)\right]$. Why do you think the FEM solution overestimates $R$ ? Can you think of a way to improve the solution to the free boundary problem? If so implement it and plot $t$ vs $R(t)$ for $t \in[0,5]$ for this implementation.

