1. Mathematics 406, Assignment 4 Due October 27th

1. (a) Express \( \delta(e^x - 2) \) as a \( \delta \)-function.
(b) Compute \( \int_0^\infty \delta'(x^2 - 1)x^2 \, dx \)
(c) Show that \( \frac{d^4|x|^3}{dx^4} = 12 \delta(x) \).
(d) Express \( \delta(x^2 - x - 6) \) as a sum of \( \delta \)-functions

2. In class we discussed the discrete analogue of the Green’s function in terms of the matrix inverse. Consider the matrix obtained by discretizing the differential operator

\[
Lu = u'' = f \quad \text{subject to } u(0) = 0 = u(1)
\]

by finite differences using \( n = 5 \) intervals, namely

\[
A_5 = \begin{bmatrix}
-2 & 1 & 0 & 0 \\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 1 & -2
\end{bmatrix}
\]

(a) Show that the constant function \( u_i = c \) and the linear function \( u_i = i \) both yield zero when multiplied by a complete row \([\ldots 0 1 -2 1 0 \ldots]\) of the matrix \( A \).

(b) Show that the ramp function \( r_{ij} = \begin{cases} 0 & \text{when } i \leq j \\ i - j & \text{when } i > j \end{cases} \) yields \( \delta_{ij} \) when multiplied by a complete row \([\ldots 0 1 -2 1 0 \ldots]\) of \( A \).

(c) Now construct a formula for the “Green’s function” for \( A_N \) (equivalently the matrix inverse \( A_N^{-1} \)) by considering the solution of \( \sum_k A_{N,ik} G_{kj} = \delta_{ij} \). Use the facts from (a) and (b) and the boundary conditions to construct the Green’s function in much the same way as is done for the continuous problem. Compare your results with the numerical inverse for \( A_5 \) obtained using MATLAB.

3. Consider the boundary value problem

\[
Lu = x^2u'' + xu' - 4u = f(x) \quad \text{(1)}
\]
\[
u(1) = 1 \quad \text{and } u(2) = 0
\]

(a) Determine the adjoint operator \( L^* \) and appropriate boundary conditions associated with \( L \) and the boundary conditions above. Determine the Green’s function \( G \) so that \( u \) has an integral representation in terms of \( G \) and \( f \).

(b) Multiply the equation in (1) by an appropriate function to make \( L \) formally self-adjoint. Now determine the Green’s function for the new problem and use this to obtain an integral representation for \( u \) in terms of \( f \). How does this compare to the integral representation found in part (a)?
4. Consider the boundary value problem

\[ Lu = u'' + k^2 u = f(x), \quad u(0) = 1 \text{ and } u(1) = 0 \]  \hspace{1cm} (2)

(a) Assuming that \( k \neq n\pi \), \( n = 0, 1, \ldots \) determine the Green’s function and obtain an integral representation for \( u \) in terms of \( f \).

(b) Show that in the limit \( k \to 0 \) the Green’s function reduces to that derived in Lecture 11.

(c) Use an eigenfunction expansion to determine \( G(s, x) \). What happens if \( k = n\pi \)? Under what condition does a solution exist?

(d) For \( k = n\pi, \ n = 1, 2, \ldots \) determine a modified Green’s function and an integral representation for \( u \) in terms of \( f \).