## 1 Mathematics 406, Assignment 3 Due October 19 th

1. (a) Express $\delta\left(e^{-x}-4\right)$ as a $\delta$-function.
(b) Compute the generalized derivative of $e^{x} \delta^{\prime}(x+3)-H(x) \cos ^{2} x$
(c) Compute the repeated generalized derivative $\frac{d^{3}|x|^{3}}{d x^{3}}$.
(d) Express $\delta\left(x^{2}+x-2\right)$ as a sum of $\delta$-functions.
(e) Define the generalized function $h(x)=\left(\frac{1}{x}, \phi\right)=P V \int_{-\infty}^{\infty} \frac{\phi(x)}{x} d x=\lim _{\epsilon \rightarrow 0}\left(\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} d x+\int_{\epsilon}^{\infty} \frac{\phi(x)}{x} d x\right)$.

Here $P V \int_{-\infty}^{\infty}$ is known as the Cauchy Principal Value integral.
i. Show that $h(x)=\int_{0}^{\infty} \frac{\phi(x)-\phi(-x)}{x} d x$
ii. Now show that the generalized derivative of $g(x)=\log |x|$ is $h(x)$.
2. In class we discussed the discrete analogue of the Green's function in terms of the matrix inverse. Consider the matrix obtained by discretizing the differential operator

$$
L u=u^{\prime \prime}=f \text { subject to } u(0)=0=u(1)
$$

by finite differences using $n=5$ intervals, namely

$$
A_{5}=\left[\begin{array}{cccc}
-2 & 1 & 0 & 0 \\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 1 & -2
\end{array}\right]
$$

(a) Show that the constant function $u_{i}=c$ and the linear function $u_{i}=i$ both yield zero when multiplied by a complete row $[\ldots 01-210 \ldots]$ of the matrix $A$.
(b) Show that the ramp function $r_{i j}=\left\{\begin{array}{c}0 \text { when } i \leq j \\ i-j \text { when } i>j\end{array}\right.$ yields $\delta_{i j}$ when multiplied by a complete row $[\ldots 01-210 \ldots$ ] of $A$.
(c) Now construct a formula for the "Green's function" for $A_{N}$ (equivalently the matrix inverse $A_{N}^{-1}$ ) by considering the solution of $\sum_{k} A_{N, i k} G_{k j}=\delta_{i j}$. Use the facts from (a) and (b) and the boundary conditions to construct the Green's function in much the same way as is done for the continuous problem. Compare your results with the numerical inverse for $A_{5}$ obtained using MATLAB.
3. Consider the boundary value problem

$$
\begin{align*}
L u & =x^{2} u^{\prime \prime}+4 x u^{\prime}+2 u=f(x)  \tag{1}\\
u(1) & =0=u(2)
\end{align*}
$$

(a) Determine the adjoint operator $L^{*}$ and appropriate boundary conditions associated with $L$ and the boundary conditions above. Determine the Green's function $G$ so that $u$ has an integral representation in terms of $G$ and $f$ and determine the value of $u$ using the integral represntation for the specific case $f=x$.
(b) Multiply the equation in (1) by an appropriate function to make $L$ formally self-adjoint. Now determine the Green's function for the new problem and use this to obtain an integral representation for $u$ in terms of $f$. How does this compare to the integral representation found in part (a)?
4. Consider the transverse motion of waves on a string that is tied down at its endpoints. The motion is governed by the one dimensional wave equation:

$$
\frac{\partial^{2} u}{\partial t^{2}}=c^{2} \frac{\partial^{2} u}{\partial x^{2}}
$$

(a) Now assume that the string is subject to harmonic forcing of the form $F(x, t)=f(x) e^{-i \omega t}$. Show that the problem reduces to the boundary value problem

$$
\begin{equation*}
L u=u^{\prime \prime}+k^{2} u=f(x), u(0)=0 \text { and } u(1)=0 \tag{2}
\end{equation*}
$$

where $k=\omega / c$ is the wave number having dimensions $m^{-1}$ and $\omega$ is the frequency having dimensions $s^{-1}$.
(b) Assuming that $k \neq n \pi \quad n=0,1, \ldots$ determine the Green's function and obtain an integral representation for $u$ in terms of $f$.
(c) Show that in the limit $k \rightarrow 0$ the Green's function reduces to that derived in Lecture 11.
(d) Use an eigenfunction expansion to determine $G(s, x)$. What happens if $k=n \pi$ ? Under what condition does a solution exist?
(e) For $k=n \pi, n=1,2, \ldots$ determine a modified Green's function and an integral representation for $u$ in terms of $f$.

