1 Mathematics 406, Assignment 2’17 Due Oct 4 th

1. Numerical Integration:
Write routines to perform numerical integration of a user-defined function \( f(x) \) over an interval \([a, b]\): \( I = \int_{a}^{b} f(x)\,dx \) by means of: the Midpoint rule with \( N \) cells, the Trapezium rule with \( N \) cells, Simpson’s rule with \( 2N \) cells and three point Gauss-Legendre quadrature in each of \( N \) cells. Use these routines to evaluate the integrals (a)-(e) below. In each case complete a table of the form:

<table>
<thead>
<tr>
<th>( h )</th>
<th>Midpoint</th>
<th>Trapezium</th>
<th>Simpson</th>
<th>Gauss-Legendre</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi/2 )</td>
<td></td>
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<tr>
<td>( \pi/4 )</td>
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<td>( \pi/16 )</td>
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<tr>
<td>( \pi/32 )</td>
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</tbody>
</table>

and determine the rate of convergence of the algorithms by calculating the error for different values of \( N \). Compare the performance of the different algorithms for each of the given integrals by establishing the order of the error term. Using the error estimate derived in class try to explain the different error behaviour for each of integrands.

(a) \( I = \int_{0}^{\pi} x \cos (3x)\,dx = -\frac{2}{9} \)
(b) \( I = \int_{0}^{\pi} \sin^4 2x\,dx = \frac{3}{8}\pi \)
(c) \( I = \int_{0}^{1} x^{3/2}\,dx = \frac{2}{5} \)
(d) \( I = \int_{0}^{1} x^{1/2}\,dx = \frac{2}{3} \)
(e) \( I = \int_{0}^{1} (-\ln x)^{5/2}\,dx = \frac{15}{5} \pi^{1/2} \) (use only the open integration rules i.e. Midpoint and Gauss integration for this integral)

2. Consider the integral, which by Cauchy’s integral formula, is given by
\[
\cos a = \frac{1}{2\pi i} \oint_{\gamma} \frac{\cos z}{z-a} \,dz
\]
where \( \gamma = \partial B_r(b) \) is a circle with radius \( r \) and centre located at \( z = b \). Use the change of variables \( z = re^{i\phi} + b \) to arrive at a 1D integral, and evaluate it using Gauss-Legendre quadrature. Use \( N = 10 \) and 20 intervals, 3 points per interval, \( a = i\pi/4, b = 1 + i \) and \( r = 3 \). Now repeat this process with the Composite Trapezium Rule. Compare the results to \( \cos a \).

3. Repeated Richardson extrapolation applied to the Trapezium Rule is known as Romberg integration. Use the following asymptotic expansion for the error in the Trepeziun rule:
\[
I(0) - I(h_s) = \sum_{i=1}^{\infty} c_i h_s^{2i}
\]
to derive the recursion for Richardson extrapolants for \( I \) in which \( h_{s+1} = \frac{1}{2}h_s \). If we define \( a_s^{(1)} = I(h_s) \) then by eliminating the \( O(h^2) \) term in the expansion show that:

\[
I(0) - \left\{ a_{s+1}^{(1)} + \frac{a_{s+1}^{(1)} - a_s^{(1)}}{2^2 - 1} \right\} = \sum_{i=2}^{\infty} c_i^{(2)} h_s^{2i}
\]

where \( c_i^{(2)} = \frac{\alpha}{3} \left( \frac{1}{(4i-1)} - 1 \right) \). Now define

\[
a_s^{(2)} = a_{s+1}^{(1)} + \frac{a_{s+1}^{(1)} - a_s^{(1)}}{2^2 - 1}
\]

and use expressions for \( a_s^{(2)} \) and \( a_{s+1}^{(2)} \) to eliminate the \( O(h^4) \) term. Now generalize this to obtain the following recursion for \( a_s^{(m)} \) in terms of \( a_s^{(m-1)} \):

\[
a_s^{(m)} = a_{s+1}^{(m-1)} + \frac{a_{s+1}^{(m-1)} - a_s^{(m-1)}}{4^{m-1} - 1}
\]

Modify the routine `trapez` posted on the course web site to perform repeated Richardson Extrapolation until a prescribed tolerance is reached. Call this new function

\[
[Integral, I, X] = \text{Romberg}(f, a, b, \text{tol}, \text{kmax})
\]

where the inputs are \( f= \) the integrand, \([a, b]= \) the domain of integration, \( \text{tol} = \) the specified tolerance, and \( \text{kmax} = \) the maximum number of refinements. The outputs are: \( \text{Integral} = \) the approximate integral, \( I= \) the table of extrapolated values, and \( X= \) the sample points. Use this routine to evaluate the integral

\[
I = \int_0^1 (1 + x^2)^{-1} dx = \frac{1}{4\pi}
\]

How many refinements are required to obtain 5 digits of precision?