1 Mathematics 406, Assignment 2’15 Due Oct 9 th

1. Spline interpolation and numerical differentiation

Consider interpolating a function \( f(x) \) at three points \( \{x_1, x_2, x_3\} \) at which the function assumes values \( \{f_1 = f(x_1), f_2 = f(x_2), f_3 = f(x_3)\} \) by two piecewise cubic polynomials \( C_1(x) \) on \([x_1, x_2]\) and \( C_2(x) \) on \([x_2, x_3]\). The two cubic polynomials must be continuous and have continuous first and second derivatives at the common node \( x_2 \). Assume the following expressions for \( C_1''(x) = \frac{d^2}{dx^2}C_1(x) \) and \( C_2''(x) \) in terms of the piecewise linear basis functions

\[
C_1''(x) = \frac{s_1''}{h_1} (x_2 - x) + \frac{s_2''}{h_1} (x - x_1) \quad \text{for} \quad x \in [x_1, x_2]
\]

\[
C_2''(x) = \frac{s_2''}{h_2} (x_3 - x) + \frac{s_3''}{h_2} (x - x_2) \quad \text{for} \quad x \in [x_2, x_3]
\]

where \( h_1 = x_2 - x_1 \) and \( h_2 = x_3 - x_2 \). By imposing the continuity of \( C \) and and continuity of its first and second derivatives at \( x_2 \) and the interpolation conditions

\[
C_1(x_1) = f_1, \quad C_1(x_2) = f_2 = C_2(x_2), \quad C_2(x_3) = f_3
\]

show that

\[
h_1s_1'' + 2(h_1 + h_2)s_2'' + h_2s_3'' = 6 \left( \frac{f_3 - f_2}{h_2} - \frac{f_2 - f_1}{h_1} \right)
\]

For the function \( f(x) = e^{-x} \) and \( \{x_1 = 0, x_2 = 0.5, x_3 = 1\} \) use the ‘natural piecewise cubic spline’ with \( s_1'' = 0 = s_3'' \) to approximate \( f'(1/2) \) and \( f''(1/2) \). Complete the following table

<table>
<thead>
<tr>
<th>( f'(1/2) )</th>
<th>( f''(1/2) )</th>
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<tbody>
<tr>
<td>( C'(1/2) )</td>
<td>( C''(1/2) )</td>
</tr>
</tbody>
</table>

Plot \( f(x) \) and \( C(x) \) i.e.: \( C_1(x) \) on \([x_1, x_2]\) and \( C_2(x) \) on \([x_2, x_3]\), and \( f'(x) \) and \( C'(x) \) i.e.: \( C_1'(x) \) on \([x_1, x_2]\) and \( C_2'(x) \) on \([x_2, x_3]\).

2. Numerical Integration:

Write routines to perform numerical integration of a user-defined function \( f(x) \) over an interval \([a, b]\): \( I = \int_a^b f(x)dx \) by means of: the Midpoint rule with \( N \) cells, the Trapezium rule with \( N \) cells, Simpson’s rule with \( 2N \) cells and three point Gauss-Legendre quadrature in each of \( N \) cells. Use these routines to evaluate the following integrals:

(a) \( I = \int_0^1 \sin 10x dx = \frac{1}{10}(1 - \cos 10) \)

(b) \( I = \int_0^\pi \sin^4 2x dx = \frac{3}{8}\pi \)

(c) \( I = \int_0^2 x^{1/4} dx = 3 \cdot 2^{-\frac{5}{2}} \)

(d) \( I = \int_0^1 (- \ln x)^{\frac{3}{2}} dx = \frac{1}{2}\pi^\frac{3}{2} \) (use only the open integration rules i.e. Midpoint and Gauss integration for this integral)
In each case determine the rate of convergence of the algorithms by calculating the error for different values of $N$. Compare the performance of the different algorithms for each of the given integrals by establishing the order of the error term. Try to explain why a given method can have different error bounds for different integrands.

3. Consider the integral, which by Cauchy’s integral formula, is given by

$$\sin a = \frac{1}{2\pi i} \oint_{\gamma} \frac{\sin z}{z - a} \, dz$$

where $\gamma = \partial B_r(b)$ is a circle with radius $r$ and centre located at $z = b$. Use the change of variables $z = re^{i\phi} + b$ to arrive at a 1D integral, and evaluate it using Gauss-Legendre quadrature. Use $N = 10$ and 20 intervals, 3 points per interval, $a = i\pi/4$, $b = 1 + i$ and $r = 3$. Now repeat this process with the Composite Trapezium Rule. Compare the results to $\sin a$. 
