1  Math 406: Ass. 1: Due Wednesday 20 Sept.

1. Least squares fitting - $m$–th degree polynomial through $N$ points.
   (a) Find a system of equations for the coefficients of an $m$–th degree polynomial that fits a function $f$ at given data points $(x_k, f(x_k)), k = 1, \ldots, N$ in the least squares sense. Write the equations in matrix form. What is the size of the matrix?
   (b) Solve the system of equations using Matlab for $x_k = \{0, 0.1, 0.3, 0.5, 0.8, 1, 1.2, 1.4, 1.6, 2\}$, $f(x) = xe^{-x}$, and $m = 5$. Evaluate the resulting polynomial at $x = 0.7$. Compare the answer to the “exact” value of $0.7e^{-0.7}$. Use the MATLAB function “polyfit” to check the results.

2. Lagrange interpolating polynomial.
   (a) Plot the Lagrange interpolating basis functions $l_1^{(3)}(x)$ and $l_3^{(3)}(x) (0 < x < 2)$, generated using $x_k = \{0, 0.1, 0.3, 0.5, 0.8, 1, 1.2, 1.4, 1.6, 2\}$. Use markers to highlight points $x_k$ on the plots.
   (b) Plot the Lagrange interpolating polynomial $p_9(x)(0 < x < 2)$ that passes through the points $(x_k, f(x_k)), f(x) = xe^{-x}$. Evaluate the resultant polynomial at $x = 0.7$. Compare the answer to the result in problem 1.

3. Finite Difference Tables: Let $S_k^N$ denote the sum of the $k$ th powers of the first $N$ integers i.e.:

$$ S_k^N = N \sum_{i=1}^{N} i^k $$

Write a simple MATLAB program to evaluate these sums for a specified value of $k$ for values of $N$ from $1 \ldots k + 3$. Now write MATLAB code to form the forward difference table (since the sample points are uniform). Notice that for each value of $k$ the difference table terminates - why does this happen? For the special case $k = 3$ extract the differences from your table and use the Gregory-Newton divided difference formula to derive the formula:

$$ S_3^N = N \sum_{i=1}^{N} i^3 = \frac{1}{4} N^2 (N + 1)^2 $$

4. The barycentric formula for Lagrange Interpolation: Let $\omega(x) = \prod_{k=0}^{N} (x - x_k)$. Show that the Lagrange interpolating polynomial that interpolates $f$ at the points $x_0, x_1, \ldots, x_N$ can be expressed as

$$ p_N(x) = \omega(x) \sum_{k=0}^{N} \frac{f_k}{(x - x_k) \omega'(x_k)} $$
Now by considering the function $f(x) \equiv 1$ derive an expression for the reciprocal of $\omega(x)$ and use it to derive the barycentric form of the Lagrange interpolating polynomial

$$p_N(x) = \frac{\sum_{k=0}^{N} \frac{w_k}{(x-x_k)} f_k}{\sum_{k=0}^{N} \frac{w_k}{(x-x_k)}}$$

where $w_k = 1/\omega'(x_k)$.

5. Spline interpolation and numerical differentiation: Consider interpolating a function $f(x)$ at three points $\{x_1, x_2, x_3\}$ at which the function assumes values $\{f_1 = f(x_1), f_2 = f(x_2), f_3 = f(x_3)\}$ by two piecewise cubic polynomials $C_1(x)$ on $[x_1, x_2]$ and $C_2(x)$ on $[x_2, x_3]$. The two cubic polynomials must be continuous and have continuous first and second derivatives at the common node $x_2$. Assume the following expressions for $C''_1(x) = \frac{d^2}{dx^2}C_1(x)$ and $C''_2(x)$ in terms of the piecewise linear basis functions

\[
C''_1(x) = \frac{s''_1}{h_1}(x_2-x) + \frac{s''_2}{h_1}(x-x_1) \quad \text{for} \quad x \in [x_1, x_2]
\]

\[
C''_2(x) = \frac{s''_3}{h_2}(x_3-x) + \frac{s''_2}{h_2}(x-x_2) \quad \text{for} \quad x \in [x_2, x_3]
\]

where $h_1 = x_2 - x_1$ and $h_2 = x_3 - x_2$. By imposing the continuity of $C$ and and continuity of its first and second derivatives at $x_2$ and the interpolation conditions

\[C_1(x_1) = f_1, \quad C_1(x_2) = f_2 = C_2(x_2), \quad C_2(x_3) = f_3\]

show that

\[
h_1s''_1 + 2(h_1 + h_2)s''_2 + h_2s''_3 = 6 \left\{ \frac{f_3 - f_2}{h_2} - \frac{f_2 - f_1}{h_1} \right\} \]

For the function $f(x) = e^{-x}$ and $\{x_1 = 0, x_2 = 0.5, x_3 = 1\}$ use the ‘natural piecewise cubic spline’ with $s''_1 = 0 = s''_3$ to approximate $f'(1/2)$ and $f''(1/2)$. Complete the following table

<table>
<thead>
<tr>
<th>$\frac{d}{dx}f'(1/2)$</th>
<th>$\frac{d}{dx}f(1/2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C''(1/2)$</td>
<td>$C'(1/2)$</td>
</tr>
</tbody>
</table>

Plot $f(x)$ and $C(x)$ i.e.: $C_1(x)$ on $[x_1, x_2]$ and $C_2(x)$ on $[x_2, x_3]$, and $f'(x)$ and $C'(x)$ i.e.: $C_1'(x)$ on $[x_1, x_2]$ and $C_2'(x)$ on $[x_2, x_3]$. 

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