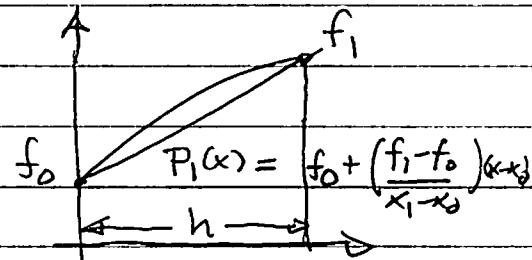


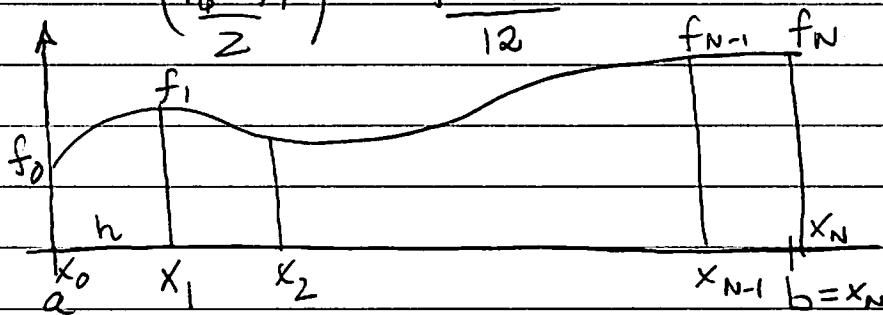
MATH 406 MIDLTERM 2013

$$1(a) \quad f(x) = P_N(x) + f^{(N+1)}(\xi)(x-x_0)\dots(x-x_N)$$

$$f(x) = f_0 + \left( \frac{f_1 - f_0}{x_1 - x_0} \right) (x - x_0) + \frac{f^{(2)}(\xi)}{2} (x - x_0)(x - x_1)$$



$$\begin{aligned} \int_{x_0}^{x_1} f(x) dx &= \int_{x_0}^{x_1} \left\{ f_0 + \left( \frac{f_1 - f_0}{x_1 - x_0} \right) (x - x_0) \right\} dx + \frac{f^{(2)}(\xi)}{2} \int_{x_0}^{x_1} (x - x_0)(x - x_1) dx \\ &= f_0(x_1 - x_0) + \left( \frac{f_1 - f_0}{x_1 - x_0} \right) \frac{(x_1 - x_0)^2}{2} + \frac{f^{(2)}(\xi)}{2} \int_{x_0}^{x_1} (x - x_0)(x - x_1) dx \\ &= (x_1 - x_0) \left[ \frac{f_1 + f_0}{2} \right] + \frac{f^{(2)}(\xi)}{2} h^3 \left[ \frac{s^3}{3} - \frac{s^2}{2} \right]_0^1 \\ &= h \left( \frac{f_0 + f_1}{2} \right) - \frac{f^{(2)}(\xi)}{12} h^3 \end{aligned}$$

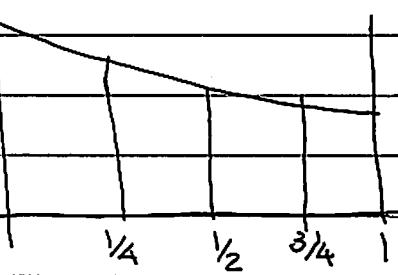


$$\begin{aligned} \int_a^b f(x) dx &= \frac{h}{2} \left[ (f_0 + f_1) + (f_1 + f_2) + \dots + (f_{N-1} + f_N) \right] - \frac{h^3}{12} \sum_{n=1}^N f^{(2)}(\xi_n) h \\ &= \frac{h}{2} \left[ f_0 + 2f_1 + \dots + 2f_{N-1} + f_N \right] - \frac{h^2}{12} \left( \sum_{n=1}^N f^{(2)}(\xi_n) h \right) \end{aligned}$$

NOW  $\sum_{n=1}^N f^{(2)}(\xi_n) h \approx \int_a^b f''(x) dx$  SINCE IT IS A RIEMANN SUM

$$\therefore \int_a^b f(x) dx = \frac{h}{2} \left[ f_0 + 2f_1 + \dots + 2f_{N-1} + f_N \right] - \frac{h^2}{12} [f'(b) - f'(a)] + O(h^4)$$

$$b) I(0) = \int_0^1 e^{-x} dx = 1 - e^{-1} = 0.632121$$



$$I\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)}{2} \left[ 1 + 2e^{-1/2} + e^{-1} \right]$$

$$= 0.645235$$

$$I\left(\frac{1}{4}\right) = \frac{\left(\frac{1}{4}\right)}{2} \left[ 1 + 2e^{-1/4} + 2e^{-1/2} + 2e^{-3/4} + e^{-1} \right]$$

$$= 0.635409$$

$$c) I(h) = I(0) + Ch^2 \quad (1) \text{ By (a)}$$

$$I\left(\frac{h}{2}\right) = I(0) + Ch^2/4 \quad (2)$$

$$4*(2) - (1) \Rightarrow 4I\left(\frac{h}{2}\right) - I(h) = 3I(0) + O(h^4)$$

$$\therefore I(0) = \frac{4}{3} I\left(\frac{h}{2}\right) - \frac{1}{3} I(h) + O(h^4).$$

$$= \frac{4}{3} 0.635409 - \frac{1}{3} 0.645235$$

$$= 0.63213.$$

$$Q2 \quad L_u = x^2 u'' - x u' - 3u = f(x) \quad u(0) < \infty \quad u(1) = 4$$

a) DIRECTLY,

$$\begin{aligned} (v, Lu) &= \int_0^1 v \{ x^2 u'' - x u' - 3u \} dx \\ &= [vx^2 u' - vx u]_0^1 - \int_0^1 u' (x^2 v)' - u (x v)' + 3uv dx \\ &= [vx^2 u' - vx u - (x^2 v)' u]_0^1 + \int_0^1 u \{ (x^2 v)'' + (x v)' - 3v \} dx \\ &= [vx^2 u']_0^1 - [vx u]_0^1 - [2x v u]_0^1 - [x^2 v' u]_0^1 + (u, L^* v) \end{aligned}$$

IF WE CHOOSE  $v(0) < \infty$  &  $v(1) = 0$  THEN ALL THE UNKNOWN TERMS INVOLVING  $u'$  DROP OUT AND WE OBTAIN

$$(v, Lu) = -4v'(1) + (u, L^* v).$$

$$\text{NOW } L^* v = (x^2 v)'' + (x v)' - 3v = (x^2 v'' + 4x v' + 2v) + (x v' + v) - 3v \\ = x^2 v'' + 5x v' = 0$$

$$\text{HOMOGENEOUS EQ } L^* v = 0 \quad v = x^r \Rightarrow r(r-1) + 5r = r(r+4) = 0 \quad r=0, -4$$

NOW THE GREEN'S FUNCTION MUST SATISFY  $L_g G(s, x) = \delta(s-x)$   $G(0, x) < \infty$   $G(1, x) = 0$

$$\text{LET } G(s, x) = \begin{cases} A_- & 0 < s < x \\ A_+ + B_+ s^{-4} & x < s < 1 \end{cases} \quad \leftarrow \begin{array}{l} \text{SINCE } G(0, x) < \infty \\ \text{DROP } s^{-4} \text{ TERM} \end{array}$$

$$0 = A_+ + B_+ \Rightarrow B_+ = -A_+$$

$$G(s, x) = \begin{cases} A_- & 0 < s < x \\ A_+ (1 - s^{-4}) & x < s < 1 \end{cases} \quad G_s = \begin{cases} 0 & 0 < s < x \\ A_+ 4s^{-5} & x < s < 1 \end{cases}$$

$$\text{CONTINUITY: } G(x_+, x) = A_- = G(x_-, x) = A_+ (1 - x^{-4}) \quad A_- = A_+ (1 - x^{-4})$$

$$\text{JUMP: } \int_{x-\epsilon}^{x+\epsilon} (s^2 G)'' + (s G)' - 3G ds = 1$$

$$\therefore \left[ s^2 G_s + 2s G \right]_{x-\epsilon}^{x+\epsilon} + \left[ s G \right]_{x-\epsilon}^{x+\epsilon} - 3 \int_{x-\epsilon}^{x+\epsilon} G ds = 1$$

$$\epsilon \rightarrow 0 \quad x^2 [G_s(x_+, x) - G_s(x_-, x)] = x^2 [A_+ 4x^{-5} - 0] = 4x^{-3} A_+ = 1 \quad A_+ = x^3/4$$

$$\therefore A_- = \frac{1}{4} (x^3 - x^{-1})$$

$$\therefore G(s, x) = \begin{cases} \frac{1}{4} (x^3 - x^{-1}) & 0 < s < x \\ \frac{x^3}{4} (1 - s^{-4}) & \end{cases}$$

$$G_s(1, x) = \frac{x^3}{4} \cdot 4 (1)^{-5} = x^3.$$

$$\therefore u(x) = 4x^3 + \frac{1}{4} (x^3 - x^{-1}) \int_0^x f(s) ds + \frac{x^3}{4} \int_x^1 (1 - s^{-4}) f(s) ds.$$

b) CONVERT TO A SELF ADJOINT PROBLEM

$$a_0 = x^2 \quad a_1 = -x \quad F(x) = \frac{e^{\int \frac{a_1}{a_0} dx}}{a_0} = \frac{e^{-\int \frac{x}{x^2} dx}}{x^2} = x^{-3}.$$

$$\therefore L u = F(L u) = x^{-1} u'' - x^{-2} u' - 3u = (x^{-1} u')' - 3u = x^{-3} f(x) = g(x)$$

NOW  $(v, L u) = \int_0^1 [v(x^{-1} u')' - 3v] dx = [v x^{-1} u' - (x^{-1} v') u] \Big|_0^1 + \int_0^1 u L v dx.$

CHOOSE  $v(0) < \infty \quad v(1) = 0 \quad \text{THEN}$

$$(v, L u) = -4v'(1) + (u, L v).$$

THE GREEN'S FUNCTION SHOULD SATISFY

$$L_s G(s, x) = \delta(s-x)$$

$$G(0, x) = 0 = G(1, x)$$

$$\text{HOMOGENEOUS EQ: } s^2 v'' - sv' - 3v = 0 \quad |$$

$$v = s^r \Rightarrow r(r-1) - r - 3 = r^2 - 2r - 3 = (r-3)(r+1) = 0 \quad r = 3, -1$$

$$\text{LET } G(s, x) = \begin{cases} A_- s^3 & 0 < s < x \\ A_+ (s^3 - s^{-1}) & x < s < 1 \end{cases}$$

$$w_0(s) = s^3 \quad w_1(s) = s^3 - s^{-1} \quad p(s) = s^{-1}$$

$$W(s) = \begin{vmatrix} s^3 & s^3 - s^{-1} \\ 3s^2 & 3s^2 + s^{-2} \end{vmatrix} = 3s^5 + s - 3s^5 + 3s = 4s$$

$$\therefore p(s) W(s) = s^{-1}(4s) = 4.$$

$$\therefore G(s, x) = \frac{1}{4} \begin{cases} (x^3 - x^{-1}) s^3 & 0 < s < x \\ (s^3 - s^{-1}) x^3 & x < s < 1 \end{cases}$$

$$\therefore u(x) = 4 G_s(1, x) + \frac{1}{4} (x^3 - x^{-1}) \int_0^x s^3 \left( \frac{f(s)}{s^3} \right) ds + \frac{x^3}{4} \int_x^1 (s^3 - s^{-1}) \frac{f(s)}{s^3} ds.$$

$$G_s(s, x) = \frac{1}{4} \begin{cases} (x^3 - x^{-1}) 3s^2 & 0 < s < x \\ (3s + s^{-2}) x^3 & x < s < 1 \end{cases} \Rightarrow G_s(1, x) = \frac{1}{4} \cdot 4 \cdot x^3$$

$$\therefore u(x) = 4x^3 + \frac{(x^3 - x^{-1})}{4} \int_0^x f(s) ds + \frac{x^3}{4} \int_x^1 (1 - s^{-4}) f(s) ds$$