Math 406, Midterm
5 November 2012

Instructions. The duration of the exam is 55 minutes. Answer all questions. A single page of A4 notes written on both sides of the page is permitted into the exam.

Maximum score 50.

1. Gauss-Hermite quadrature with $m = 2$ integration points evaluates integrals of the form

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx$$

exactly if $f(x)$ is a polynomial of degree $2m - 1 = 3$. Use this fact to determine the integration points ξ_i and weights w_i in the formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \simeq w_1 f(\xi_1) + w_2 f(\xi_2)$$

To simplify your calculation you may assume the symmetry conditions $w_1 = w_2$ and $\xi_1 = -\xi_2$.

Hint: It may also be helpful to know that $\int_{-\infty}^{\infty} e^{-x^2} dx = \pi^{1/2}$ and $\int_{-\infty}^{\infty} e^{-x^2} x^2 dx = \frac{1}{2}\pi^{1/2}$

[10 marks]

2. Consider the boundary value problem

$$\begin{aligned} Lu &= u'' + \mu^2 u = f(x) && (1) \\ u'(0) &= 0 = u'(1) && (2) \end{aligned}$$

(a) Find the Green's function $G(s, x)$ for the boundary value problem (1)-(2) and use $G(s, x)$ to express the solution $u(x)$ to (1)-(2) as an integral involving $f(x)$.

[10 marks]

(b) For what values of μ is it not possible to solve the boundary value problem (1)-(2)? For each of these values of μ, determine the solvability condition on f so that it is possible to find such a solution.

[5 marks]

(c) Determine the modified Green's function when $\mu = 0$.

[10 marks]

(d) Determine the weak statement of (1)-(2) for the special case $f(x) = 0$, in which (1)-(2) reduces to an eigenvalue problem. Now determine a Galerkin approximation to this weak form using a single finite element and two piecewise linear finite element basis functions $N_i(x)$. Note that on this domain these basis functions assume a particularly simple form. Determine the stiffness and mass matrices for the approximation and estimate the first two eigenvalues by setting an appropriate determinant to zero. Compare these to the exact eigenvalues.

[15 marks]