Be sure that this examination has 5 Questions and 17 pages including this cover

The University of British Columbia

Final Examinations - 17 April 2012

Mathematics 257/316

All Sections

Time: 2.5 hours

First Name
(USE CAPITALS) ________________
Signature _______________________
Student Number ___________________

Last Name
(USE CAPITALS) ________________
Instructor's Name ___________________
Section Number ___________________

Special Instructions:

Students are not allowed to bring any notes into the exam.
No calculators are allowed.

Rules governing examinations:

1. Each candidate should be prepared to produce his/her
   library/AMS card upon request.

2. Read and observe the following rules:
   No candidate shall be permitted to enter the examination room after
   the expiration of one half hour, or to leave during the first half hour
   of the examination.
   Candidates are not permitted to ask questions of the invigilators,
   except in cases of supposed errors or ambiguities in examination
   questions.
   CAUTION - Candidates guilty of any of the following or similar practices
   shall be immediately dismissed from the examination and shall be liable to
   disciplinary action.
   (a) Making use of any books, papers or memoranda, other than those
       authorized by the examiners.
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   (c) Purposefully exposing written papers to the view of other candidates.
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3. Smoking is not permitted during examinations.

1 20
2 20
3 20
4 20
5 20

Total 100
1. Consider the differential equation
\[ Lf = 6x^2y'' - xy' + 2(1 + x)y = 0 \]  

(a) Classify the points \( 0 \leq x < \infty \) as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of \( r \) such that there are solutions of the form \( y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \).

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

\[ a_0 = \frac{1}{0!} \left( \frac{6}{6} \right) x^{1/2} = \frac{6}{6!} x^{1/2} \]

\[ a_2 = \frac{1}{2!} \left( \frac{6}{6} \right) x^{3/2} = \frac{6}{6!} x^{3/2} \]

\[ a_4 = \frac{1}{4!} \left( \frac{6}{6} \right) x^{5/2} = \frac{6}{6!} x^{5/2} \]
2. Consider the following initial boundary value problem for the heat equation:
\[
\begin{aligned}
&u_t = u_{xx} - u, \quad 0 < x < 1, \quad t > 0 \\
&u(0, t) = 0, \quad u_x(1, t) = 1 \\
&u(x, 0) = 1
\end{aligned}
\] (2)

(a) Determine a steady state solution to the boundary value problem.

(b) Use this steady state solution to determine the solution to the boundary value problem (2) by separation of variables.

HINT: The following integral may be useful:
\[
\int_0^1 \sinh(x) \sin(\beta x) dx = \frac{1}{\beta^2 + 1} (\sin \beta \cosh 1 - \beta \cos \beta \sinh 1)
\]

(c) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to \(O(\Delta x^2, \Delta t)\) terms. Use the notation \(u_n^k \approx u(x_n, t_k)\) to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition \(u_x(1, t) = 1\) with \(O(\Delta x^2)\) accuracy.

\(\text{return the plain text representation as if you were reading it naturally.}\)

(a) Look for \(W(x)\):
\[
\begin{aligned}
W(x) &= W(0) = 0 \quad W(1) = 1 \\
W(x) &= A \cosh x + B \sinh x \quad W'(x) = A \sinh x + B \cosh x \\
W(0) &= A = 0 \quad W'(1) = B \cosh(1) = 1 \quad \Rightarrow \quad B = \frac{1}{\cosh(1)} \\
\therefore \quad W(x) &= \frac{\sinh x}{\cosh(1)}
\end{aligned}
\]

(b) Let \(u(x, t) = W(\tilde{x}) + V(x, \tilde{t})\)
\[
\begin{aligned}
\tilde{V} &= V_x - V \\
0 &= u(0, t) = W(0) + V(0, \tilde{t}) = 0 + V(0, \tilde{t}) \quad \Rightarrow \quad V(0, \tilde{t}) = 0 \\
\frac{\partial}{\partial t} V(x, \tilde{t}) &= W'(1) + V_x(1, \tilde{t}) = 1 + V_x(0, \tilde{t}) \quad \Rightarrow \quad V_x(1, \tilde{t}) = 1 \\
1 &= u(1, 0) = W(1) + V(1, 0) \quad \Rightarrow \quad V(1, 0) = 0 \\
\end{aligned}
\]

Now let \(V(x, \tilde{t}) = \frac{\sinh x}{\cosh(1)} \frac{\tilde{t}}{\tilde{t}(\tilde{x})} + 1 = \frac{\tilde{t}}{\tilde{t}(\tilde{x})} = -\lambda^2 \tilde{t}
\]
\[
\begin{aligned}
\tilde{t} &= -\left(1 + \lambda^2\right) \tilde{t} \quad \Rightarrow \quad \tilde{t}(\tilde{x}) = CE^{-\left(1 + \lambda^2\right) \tilde{t}} \\
\mathbf{X} &= \begin{cases} \mathbf{X} + \lambda^2 \mathbf{X} = 0 \\ \mathbf{X}(0) = 0 = \mathbf{X}'(1) \end{cases} \\
\lambda_n &= \left(2n + 1\right) \frac{\pi}{2} \quad \mathbf{X}_n = \sin(\lambda_n x) \quad \eta = 0, 1, \ldots \\
\therefore \quad V(x, \tilde{t}) &= \sum_{\eta = 0}^{\infty} C_n \frac{e^{-\left(1 + \lambda_n^2\right) \tilde{t}}}{\lambda_n^2} \sin(\lambda_n x) \\
\text{Now} \quad 1 - \frac{\sinh x}{\cosh(1)} = V(x, 0) = \sum_{\eta = 0}^{\infty} C_n \sin(\lambda_n x) \quad \therefore \quad C_n = \frac{2}{\lambda_n} \int_0^1 \left[1 - \frac{\sinh x}{\cosh(1)}\right] \sin(\lambda_n x) dx \frac{1}{\lambda_n^2 + 1} \left[\sin \lambda_n \cosh(1) - \lambda_n \cos \lambda_n \sinh(1)\right]
\end{aligned}
\]
\[
\begin{aligned}
C_n &= -2 \frac{\cos \lambda_n \pi}{\lambda_n} \int_0^1 \frac{1}{\lambda_n^2 + 1} \left[\sin \lambda_n \cosh(1) - \lambda_n \cos \lambda_n \sinh(1)\right] dx \frac{1}{\lambda_n^2 + 1} \\
&= \frac{2}{\lambda_n} - \frac{2}{\lambda_n^2 + 1} (-1)^n \\
\therefore \quad u(x, \tilde{t}) &= \frac{\sinh x}{\cosh(1)} + 2 \sum_{\eta = 0}^{\infty} \frac{\left[1 - (-1)^n! \right]}{\lambda_n^2 + 1} \left[\sin \lambda_n \cosh(1) - \lambda_n \cos \lambda_n \sinh(1)\right]
\end{aligned}
\]
(C) Divide the spatial interval \([0, 1]\) into \((n+1)\) mesh points such that \(x_n = n\Delta x\) and consider time steps \(t_k = k\Delta t\).

Use the first order difference approximation in time:

\[
U(x, t_{k+1}) = U(x, t_k) + \frac{\partial U(x, t_k)}{\partial t} \Delta t + \frac{1}{2!} \frac{\partial^2 U(x, t_k)}{\partial t^2} \Delta t^2
\]

\[
\therefore \quad \frac{\partial U(x_n, t_k)}{\partial t} = \frac{U_{n+1} - U_n}{\Delta t} + O(\Delta t^2)
\]

Use central differences in space:

\[
U(x+\Delta x, t) = U(x, t) + \Delta x \frac{\partial U(x, t)}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 U(x, t)}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 U(x, t)}{\partial x^3} + \frac{\Delta x^4}{4!} \frac{\partial^4 U(x, t)}{\partial x^4} + O(\Delta x^4)
\]

\[
\therefore \quad \frac{\partial U(x, t)}{\partial x} = \frac{U_{n+1} - 2U_n + U_{n-1}}{\Delta x^2} + O(\Delta x^2)
\]

\[
\therefore \quad U_t = U_{xx} - \gamma = \frac{U_{n+1} - 2U_n + U_{n-1}}{\Delta t} + \gamma \frac{U_{n+1} - 2U_n + U_{n-1}}{\Delta x^2}
\]

or \( U_n^{k+1} = \gamma U_{n+1}^{k+1} - (\Delta t + \gamma) U_n^k + \gamma U_{n-1}^k \quad \gamma = \frac{\Delta t}{\Delta x^2} \)

To obtain a 2nd order approximation \((k)\) implies:

\[
\frac{U_{n+1}^k - U_n^k}{2\Delta x} = 1 \quad \text{or} \quad U_{n+1}^k = U_n^k + 2\Delta x
\]

\(\text{Ghost mesh points}\).
3. Consider the following initial-boundary value problem:
\[ u_{tt} + 2\gamma u_t = c^2 u_{xx}, \quad 0 < x < 1 \]
\[ u(0, t) = u(1, t) = 0 \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = 0. \]

(a) Assuming that \( 0 < \gamma < \pi c \), use separation of variables to determine the solution to this boundary value problem. [12 marks]

(b) If \( \gamma = 0 \), \( c = 1 \), and if the initial displacement \( f(x) = \sin(4\pi x) \), sketch the shape of the string at time \( t = 1/4 \). [8 marks] 
[total 20 marks]

\[ u(x, t) = \sum_{n=1}^{\infty} C_n \sin \lambda_n x \]
\[ u_t(x, t) = \sum_{n=1}^{\infty} \frac{\lambda_n}{c^2} C_n \lambda_n \sin \lambda_n x \]

\[ f(x) = \sin(4\pi x) \]
\[ u(x, t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n t) \sin \lambda_n x \]

\[ A_n = \frac{2}{\lambda_n} \int_0^1 f(x) \sin \lambda_n x \, dx \]

\[ u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{\pi}{2} t \right) + \frac{\gamma}{\lambda_n} A_n \sin \left( \frac{\pi}{2} t \right) \right) \sin \lambda_n x \]

(b) \( \gamma = 1 \) \( \Rightarrow \) \( u(x, t) = \sum_{n=1}^{\infty} A_n \sin \lambda_n x \sin \lambda_n x \]
\[ A_n = \frac{2}{\lambda_n} \int_0^1 f(x) \sin \lambda_n x \, dx \]

\[ f(x) = \sin(4\pi x) \]
\[ A_n = \frac{2}{\lambda_n} \int_0^1 \sin(4\pi x) \sin \lambda_n x \, dx \]
\[ u(x, t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) \sin(\lambda_n x) \]

\[ u(x, t) = \left\{ \begin{array}{ll}
\sin(4\pi x) & \text{if } 0 < x < 1/4 \\
-\sin(4\pi x) & \text{if } 1/4 < x < 1/2 \\
\sin(4\pi x) & \text{if } 1/2 < x < 1 \\
\end{array} \right. \]
4. (a) Consider the eigenvalue problem
\[ r^2 R'' + r R' + \lambda R = 0 \]
\[ R(1) = 0 = R(2) \]
Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for the annular region:
\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi \]  
\[ u(r, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r, \pi) = f(r) \]
\[ u(1, \theta) = 0 \quad \text{and} \quad u(2, \theta) = 0 \]
[12 marks]
[total 20 marks]

(A) \[ r^2 R'' + r R' + \lambda R = 0 \quad \ldots \quad (*) \]
\[ \div \tau: -r R'' - R' = \lambda R \Rightarrow -\left(r R'\right)' = \lambda R \]

Since this is a Cauchy-Euler DE, look for a solution of the form \( R(r) = r^\xi \).
\[ \begin{align*}
\xi(\xi - 1) + \xi + \lambda &= \xi^2 + \lambda = 0 \\
\xi &= \pm i \sqrt{\lambda} = \pm i \mu \quad \text{where} \mu = \sqrt{\lambda}.
\end{align*} \]

\[ R(r) = C_1 r^\mu + C_2 r^{-\mu} = A \cos \mu \ln r + B \sin \mu \ln r \]
\[ 0 = R(1) = A \quad 0 = R(2) = B \sin \mu \ln 2 \Rightarrow \mu = \frac{n \pi}{\ln 2}, \quad n = 1, 2, \ldots \]

AND \( R_n(r) = \sin \mu_n \ln r = \sin \left( \frac{n \pi \ln r}{\ln 2} \right) \]

(b) Let \( u(r, \theta) = R(r) \Theta(\theta) \) and substitute into (*) and multiply by \( \frac{r^2}{R(r)} \Theta(\theta) : \)
\[ \frac{r^2 R'' + r R' + \mu^2 R}{R(r)} = -\frac{\Theta''}{\Theta} = -\mu^2 \]
\[ R(1) = 0 = R(2) \Rightarrow \mu_n = \frac{n \pi}{\ln 2} \]
\[ \Theta'' - \mu_n^2 \Theta = 0 \]
\[ \Theta(0) = 0 \quad \text{and} \quad \Theta(\pi) = A = 0 \]
\[ \Theta_n(\theta) = 0 \quad \Rightarrow \quad \Theta_n(\theta) = A = 0 \]

\[ u(r, \theta) = \sum_{n=1}^{\infty} \frac{B_n}{\mu_n} \sinh(\mu_n \theta) \sin(\mu_n \ln r) \]
\[ \frac{\partial u}{\partial \theta} = \sum_{n=1}^{\infty} B_n \mu_n \cosh(\mu_n \theta) \sin(\mu_n \ln r) \]
\[ f(r) = \frac{\partial u}{\partial \theta}(r, \pi) = \sum_{n=1}^{\infty} B_n \mu_n \cosh(\mu_n \pi) \sin(\mu_n \ln r) \]

Now project \( f(r) \) onto \( R_n(r) \):
\[ \int_1^2 \frac{1}{r} f(r) \sin(\mu_n \ln r) \, dr = \sum_{n=1}^{\infty} B_n \mu_n \cosh(\mu_n \pi) \int_1^2 \frac{1}{r} \sin(\mu_n \ln r) \, dr \]
\[ = \sum_{n=1}^{\infty} B_n \mu_n \cosh(\mu_n \pi) \ln 2 / 2 \]
\[ \Rightarrow \quad C_n = B_n \mu_n \cosh(\mu_n \pi) = \frac{\ln 2}{2} \int_1^2 f(r) \sin(\mu_n \ln r) \, dr \]
5. Solve the inhomogeneous heat conduction problem:

\[ u_t = u_{xx} + e^{-t}(1-x), \quad 0 < x < 1, \quad t > 0 \]
\[ u(0, t) = 1, \quad \text{and} \quad u(1, t) = e^{-t} \]
\[ u(x, 0) = 1. \]

**[20 marks]**

Let us look for a function \( W(x, t) = u(x, t) + b(x) \) that satisfies the inhomogeneous BC. \( W(0) = a(t) = 1 \)

\[ W(0) = a(t) = 1 + b(0) = e^{-t} \rightarrow b = (e^{-t} - 1) \]

\[ W(x, t) = 1 + (e^{-t} - 1)x \]

Now let \( u(x, t) = W(x, t) + V(x, t) \)

\[ u_b = W_t + V_t = -e^{-t} \rightarrow V_t = W_t + e^{-t} \]
\[ V(0, t) = W(0, t) + V(0, t) = 1 + V(0, t) \]
\[ V(1, t) = W(1, t) + V(1, t) = e^{-t} + V(1, t) \]
\[ V(x, 0) = W(x, 0) + V(x, 0) = 1 + V(x, 0) \]

Now the eigenvalues and eigenfunctions associated with the homogeneous BC are \( \lambda_n = \frac{n \pi}{L}, \quad n = 1, 2, \ldots \)
\[ \lambda_n(x) = \sin(n \pi x) \]

Now expand the source term \( e^{-t} \) in terms of eigenfunctions

\[ e^{-t} = \sum_{n=1}^{\infty} \frac{1}{2} \sin(n \pi x) \]
\[ \cdot \frac{1}{n \pi} \sin(n \pi x) \rightarrow \frac{1}{n \pi} = e^{-t} \sum_{n=1}^{\infty} \frac{1}{2} \sin(n \pi x) \]

\[ \therefore \lambda_n(t) = 2e^{-t} \frac{1}{n \pi} \left[ -\cos(n \pi x) \right]_0^L = 2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} \]

Now assume the eigenfunction expansion \( V(x, t) = \sum_{n=1}^{\infty} \frac{1}{2} \sin(n \pi x) \]
\[ \therefore V_t - V_{xx} - e^{-t} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n \pi} \sin(n \pi x) \right\} \]
\[ \cdot \frac{1}{n \pi} \left[ -\cos(n \pi x) \right]_0^L = 2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} \]
\[ \lambda_n V_n = \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} \]
\[ \therefore \frac{d}{dt} \left\{ \lambda_n \right\} V_n = 2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} \cdot \lambda_n \]
\[ \therefore V_n(t) = 2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} + C_n e^{-\lambda_n^2 t} \cdot \lambda_n \]

\[ V(x, t) = \sum_{n=1}^{\infty} \left\{ 2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} + C_n e^{-\lambda_n^2 t} \cdot \lambda_n \right\} \sin(n \pi x) \]
\[ \therefore O = V(x, 0) = \sum_{n=1}^{\infty} \left\{ 2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} + C_n e^{-\lambda_n^2 t} \cdot \lambda_n \right\} \sin(n \pi x) \rightarrow C_n = -2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] \]
\[ \therefore u(x, t) = 1 + (e^{-t} - 1)x + \sum_{n=1}^{\infty} \left\{ 2 \frac{1}{n \pi} \left[ 1 - (-1)^n \right] e^{-t} + C_n e^{-\lambda_n^2 t} \cdot \lambda_n \right\} \sin(n \pi x). \]
4. (a) Consider the eigenvalue problem
\[ r^3 R'' + r R' + \lambda R = 0 \]
\[ R(1) = 0 = R(2) \]
Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for the annular region:
\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi \]
\[ u(r, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r, \pi) = 1 \]
\[ u(1, \theta) = 0 \quad \text{and} \quad u(2, \theta) = 0 \]
[12 marks]
[total 20 marks]
(Question 4 Continued)

\[ B_n = \frac{2}{\mu_n^2 \cosh(\mu_n \pi)} \left[ -\cos \left( \frac{\mu_n \pi}{L} \right) \right] \]

\[ = \frac{2}{\mu_n^2 \cosh(\mu_n \pi)} \left[ 1 - \cos \left( \frac{\pi}{L} \right) \right] \]

\[ = \frac{2}{\mu_n^2 \cosh(\mu_n \pi)} \left[ 1 - (-1)^n \right] \]

\[ = \begin{cases} 
\frac{4}{\mu_n^2 \cosh(\mu_n \pi)} & \text{n odd} \\
0 & \text{n even} 
\end{cases} \]

\[ u(\tau, \theta) = \sum_{k=0}^{\infty} \frac{4}{\left(\frac{(2k+1)\pi}{L}\right)^2 \cosh\left(\frac{(2k+1)\pi}{L}\right)} \sinh\left(\frac{(2k+1)\pi}{L} \cdot \theta \right) \sin\left(\frac{(2k+1)\pi}{L} \cdot \tau \right) \]

\[ = \frac{4 \cdot \alpha}{\pi L} \sum_{k=0}^{\infty} \frac{\sinh\left(\frac{(2k+1)\pi}{L} \cdot \theta \right) \sin\left(\frac{(2k+1)\pi}{L} \cdot \tau \right)}{\cosh\left(\frac{(2k+1)\pi}{L}\right)} \]