1. Consider the differential equation

\[ Lg = 6x^2y'' + 5xy' - (1 + x)y = 0 \]  

(a) Classify the points \( 0 \leq x < \infty \) as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of \( r \) such that there are solutions of the form \( y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \).

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

(a) \( 0 < x < \infty \) are ordinary points \& \( x = 0 \) is a singular point

\[
\lim_{x \to 0} \frac{6}{6x^2} = \frac{1}{6} \quad \lim_{x \to 0} \frac{21}{6x} = \frac{7}{2} \quad 1/6 \& 7/2 < \infty \Rightarrow x = 0 \text{ is a regular singular pt.}
\]

(b) Indicate a relationship for which \( y' + \frac{5}{6} \eta - \frac{1}{6} = 0 \), \( 6\eta^2 - \eta - 1 = (3\eta + 1)(2\eta - 1) = 0 \), \( \eta = \frac{-1}{3}, \frac{1}{2} \).

(c) \( y = \sum_{n=0}^{\infty} a_n x^{n+r} \), \( y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \), \( y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} \)

\[
Ly = 6x^2y'' + 5xy' - (1 + x)y = 0
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{6(n+r)(n+r-1)}{6n^2} \right) a_n x^{n+r} + \sum_{n=0}^{\infty} \left( \frac{5(n+r)}{6n} \right) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{6(n+r)(n+r-1) + 5}{6n^2} \right) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{6(n+r)(n+r-1) - 1}{6n^2} \right) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\]

\[
x^{m+r}, m \neq 1
\]

\[
a_m = \frac{a_{m-1}}{(m+r)[6(m+r)-1]} = \frac{a_{m-1}}{(6m^2 - 5m + 1)}
\]

\[
t = -\frac{1}{3}: \quad a_m = \frac{a_{m-1}}{(6m^3 - m - 3)} = \frac{a_{m-1}}{3m - 1}
\]

\[
a_1 = \frac{a_0}{1} = a_0, \quad a_2 = \frac{a_1}{2.7} = \frac{a_0}{14}
\]

\[
h_1(x) = a_0 \left( 1 + x + \ldots \right)
\]

\[
t = \frac{1}{2}: \quad a_m = \frac{a_{m-1}}{(m+1)(6m+2)} = \frac{a_{m-1}}{(6m^2 + 5m + 1)}
\]

\[
a_1 = \frac{a_0}{7} = \frac{a_0}{11}, \quad a_2 = \frac{a_1}{2.17} = \frac{a_0}{22.17}
\]

\[
h_2(x) = a_0 \left[ 1 + \frac{x}{11} + \frac{x^2}{22.17} + \ldots \right]
\]
2. Consider the following diffusion initial-boundary value problem

\[ u_t = u_{xx}, \quad 0 < x < \pi/2, \quad t > 0 \]
\[ u_x(0, t) = 0 = u(\pi/2, t) \]
\[ u(x, 0) = x \]  \hfill (2)

(a) Determine the solution to (2) by separation of variables.

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to \( O(\Delta x^2, \Delta t) \) terms. Use the notation \( u_n^k \approx u(x_n, t_k) \) to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition \( u_x(0, t) = 0 \) with \( O(\Delta x^2) \) accuracy.

Hint: Taylor's expansion may prove useful: \( f(x + \Delta x) = f(x) + f'(x) \Delta x + \frac{f''(x)}{2!} \Delta x^2 + O(\Delta x^3) \).

[6 marks]

[total 20 marks]
3. Solve the following inhomogeneous initial boundary value problem for the wave equation:

\[(1) \quad u_{tt} = c^2 u_{xx} + e^{-t} \sin(3x) + 1, \quad 0 < x < \frac{\pi}{2}, \quad t > 0 \]

\[u(0,t) = t^2/2 \text{ and } u_x(\frac{\pi}{2}, t) = t, \quad t > 0 \]

\[u(x,0) = 0, \quad u_t(x,0) = \sin(5x) + x, \quad 0 < x < \frac{\pi}{2} \]

[Total 20 marks]

Find a function \( W(x,t) = A(t) x + B(t) \) that satisfies the non-zero BC

\[t^2/2 = W(0,t) = B(t) \quad \text{and} \quad t = W_x(\frac{\pi}{2}, t) = A(t) \quad \Rightarrow \quad W(x,t) = x t + t^2/2 \Rightarrow W_{tt} = x t + t^2 \Rightarrow W_{tt} = 1 \]

Now let \( u(x,t) = W(x,t) + V(x,t) \) then substitute into (1)

\[(W_{tt} + V_{tt}) = (x t + t^2/2 + V_{tt}) = C^2 (V_{xx} + V_{xx}) + e^{-t} \sin(3x) + 1 \]

BC: \[x^2/2 = V(0,t) + V(0,t) = V_{xx}(0,t) = V_{xx}(\frac{\pi}{2}, t) = \frac{\partial^2}{\partial x^2} (V_{x}(0,t) + V_{x}(\frac{\pi}{2}, t)) \]

\[V_{xx}(0,t) = \frac{\partial^2}{\partial x^2} V_x(0,t) = \frac{\partial^2}{\partial x^2} V_x(\frac{\pi}{2}, t) \]

IC: \[V(0,t) = 0 \quad \text{and} \quad V(\frac{\pi}{2}, t) = 0 \]

\[V(x,0) = 0 \quad \text{and} \quad V_t(x,0) = \sin(5x) \]

Since the BVP (2) has homogeneous BC the associated eigenfunctions are \( \mu_n = (2n+1) \pi \), \( \sin n \pi x \n \in = 0, 1, 2, \ldots \)

Now use the eigenfunction expansion:

\[e^{-t} \sin(3x) = \sum_{n=0}^{\infty} \sin(n \pi x) \sum_{n=0}^{\infty} \frac{B_n}{\mu_n} \sin(n \pi x) \]

And assume \( V(x,t) = \sum_{n=0}^{\infty} v_n(t) \sin(n \pi x) \quad V_{tt} = \sum_{n=0}^{\infty} \frac{\partial^2}{\partial t^2} v_n(t) \sin(n \pi x) \quad V_{xx} = \sum_{n=0}^{\infty} \frac{\partial^2}{\partial x^2} v_n(t) \sin(n \pi x) \]

\[0 = V_{tt} - C^2 V_{xx} - e^{-t} \sin(3x) = \sum_{n=0}^{\infty} \left( \frac{\partial^2}{\partial x^2} v_n(t) - C^2 \frac{\partial^2}{\partial x^2} v_n(t) \sin(n \pi x) \right) \sin(n \pi x) \]

Since the \( \sin(n \pi x) \) are L.I. functions \[ \frac{\partial^2}{\partial t^2} v_n(t) - C^2 \frac{\partial^2}{\partial x^2} v_n(t) = \delta_n(t) \]

The solution to the homogeneous ODE is \( v_n^h = A_n \cos(\mu_n c t) + B_n \sin(\mu_n c t) \)

For a particular solution \( v_n^p = D \cos(\mu_n c t) + E \sin(\mu_n c t) \)

Using \( v_n^h + v_n^p \) to get the general solution we obtain

\[v(t,x) = \sum_{n=0}^{\infty} A_n \cos(\mu_n c t) + B_n \sin(\mu_n c t) + \sum_{n=0}^{\infty} \left( A_n + B_n \right) \sin(2n+1) \]

Impose IC: \[0 = V(x,0) = \sum_{n=0}^{\infty} \left( A_n + \frac{B_n}{\mu_n} \right) \sin(2n+1) x \]

\[A_n = -\frac{B_n}{\mu_n^2} \]

\[B_n = \mu_n C \left[ -A_n \sin(\mu_n c t) + B_n \cos(\mu_n c t) \right] - \frac{\delta_n e^{-t}}{\mu_n^2} \sin(2n+1) \]

\[\sin(2n+1) x \Rightarrow B_n = \frac{\delta_{n1}}{\mu_n C} + \frac{\delta_{n1}}{\mu_n C^2} \]

\[v(t,x) = \sum_{n=0}^{\infty} \left( A_n \cos(\mu_n c t) + B_n \sin(\mu_n c t) + \frac{\delta_{n1} e^{-t}}{\mu_n^2} \sin(2n+1) \right) \sin(2n+1) x \]

\[u(x,t) = W(x,t) + V(x,t) = \sum_{n=0}^{\infty} \left( -\frac{\delta_{n1}}{\mu_n^2} \cos(\mu_n c t) + \left( \frac{\delta_{n2} + \delta_{n1}}{\mu_n C} \right) \sin(\mu_n c t) + \frac{\delta_{n1} e^{-t}}{\mu_n^2} \sin(2n+1) \right) \sin(2n+1) x \]

\[+ \frac{c}{1+c^2} \sin(3x) \]
4. Consider the eigenvalue problem

\[ x^2 y'' + xy' + \lambda y = 0 \quad (1) \]
\[ y(1) = 0 = y'(2) \]

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. \[ [8 \text{ marks}] \]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the quarter-annular region:

\[ u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi/2 \]
\[ u(r,0) = 0 \quad \text{and} \quad \frac{\partial u(r, \pi/2)}{\partial \theta} = f(r) \]
\[ u(1,\theta) = 0 \quad \text{and} \quad \frac{\partial u(2, \theta)}{\partial r} = 0 \]

\[ [12 \text{ marks}] \]

\[ \text{total 20 marks} \]

\[ a) \ W(x) = \frac{e^{\int \frac{x}{x^2} dx}}{x^2} = \frac{e^{\frac{\ln x}{x^2}}}{x^2} = \frac{1}{x} \quad \text{so multiply (1) by } W(x) \text{ to obtain} \]

\[ L y = -\left( x y' \right)' = \lambda \frac{x}{x^2} y \quad \text{which is in S-L form.} \]

\[ \text{Let } y(x) = x^\mu \Rightarrow y'(x) + x + \lambda = x^2 + \lambda = \frac{\mu^2}{\mu^2} = 0 \Rightarrow \mu = \pm i \lambda \quad \text{where } \lambda = \mu^2 \]

\[ \therefore \ \mu(x) = A \cos(\lambda x) + B \sin(\lambda x) \quad \mu'(x) = -A \lambda \sin(\lambda x) + B \lambda \cos(\lambda x) \]
\[ \mu(1) = A = 0 \Rightarrow \mu'(2) = \frac{B \lambda}{2} \cos(\lambda x) \Rightarrow \mu_n = \frac{(2n+1)\pi}{2} \quad n = 0, 1, \ldots \] are eigenvalues.

\[ b) \text{ Let } U(r, \theta) = R(r) \Theta(\theta) \text{ then} \]

\[ \frac{T^2 R'' + T R'}{R(r)} = -\frac{\Theta''}{\Theta(\theta)} = -\lambda = -\frac{\mu^2}{\mu^2} \text{ constant} \]

\[ \Theta(0) = 0 \quad \Theta(\pi) = 0 \Rightarrow \Theta(0) = 0 \Rightarrow \Theta = B \sin(\mu \theta). \]

\[ R(1) = 0 = R'(2) \quad \Rightarrow \mu_n = \frac{(2n+1)\pi}{2} \quad n = 0, 1, \ldots \]

\[ \therefore u(r, \theta) = \sum_{n=0}^{\infty} B_n \sin(\mu_n \theta) \sin(\mu_n \theta) \]
\[ \frac{\partial u}{\partial \theta} = \sum_{n=0}^{\infty} B_n \mu_n \cos(\mu_n \theta) \sin(\mu_n \theta) \]

\[ \text{LAST BC: } f(r) = \frac{\partial u(\gamma, \pi/2)}{\partial \theta} = \sum_{n=0}^{\infty} B_n \mu_n \cos(\mu_n \pi/2) \sin(\mu_n \theta) = \sum_{n=0}^{\infty} d_n \sin(\mu_n \theta) \]

\[ \text{Then } \int_{\gamma}^{\pi/2} f(r) \sin(\mu_n \theta) \, dr = \sum_{n=0}^{\infty} d_n \int_{\gamma}^{\pi/2} \sin(\mu_n \theta) \, d\theta = \sum_{n=0}^{\infty} d_n \left[ \frac{\cos(\mu_n \theta)}{\mu_n} \right]_{\gamma}^{\pi/2} = \sum_{n=0}^{\infty} B_n \mu_n \cos(\mu_n \pi/2) \sin(\mu_n \theta) \]

\[ \therefore \ d_m = B_n \mu_n \cos(\mu_n \pi/2) = \frac{2}{\mu_n} \int_{\gamma}^{\pi/2} f(r) \sin(\mu_n \theta) \, dr \]

\[ \therefore B_n = \frac{2}{\mu_n} \int_{\gamma}^{\pi/2} f(r) \sin(\mu_n \theta) \, dr \]
5. Solve the inhomogeneous heat conduction problem with heat loss, a time dependent source, and subject to time dependent boundary conditions:

\[ u_t = u_{xx} - u + e^{-t} \sin(x), \; 0 < x < \frac{\pi}{2}, \; t > 0 \]  
\[ u(0,t) = 0, \; \text{and} \; \frac{\partial u(\pi/2,t)}{\partial x} = e^{-t} \]
\[ u(x,0) = x. \]

Let \( W(x,t) = A(t)X + B(t) \) MATCH THE BC.

\( O = W(0,t) = B(t) \) \hspace{1cm} \( W_X = A(t) \) \hspace{1cm} \( W_X(\pi/2,t) = A(t) = e^t \)
\[ W = x e^{-t} \] \hspace{1cm} \( W_t = -xe^{-t} \)

**NOW LET \( U(x,t) = W(x,t) + V(x,t) \) AND SUBSTITUTE INTO (1)**

\[ U_t = W_t + V_t = -xe^{-t} + V_t = \left( \frac{\partial V}{\partial t} + V \right) - \left( e^{-t} V \right) + e^{-t} \sin x \]
\[ V_t = V_{xx} - V + e^{-t} \sin x \]

BC: \[ O = U(0,t) = W(0,t) + V(0,t) = 0 + V(0,t) \]
\[ \Rightarrow V_t(0,t) = V(0,t) = 0 \]
\[ \Rightarrow V(x,0) = 0 \]

IC: \[ x^2 = U(x,0) = W(x,0) + V(x,0) = x + V(x,0) \]

**SINCE \( V \) SATISFIES HOMOGENEOUS BC ASSOCIATED WITH THE EIGENVALUES EIGENFUNCTIONS**

\[ \sum_{n=0}^{\infty} \left( \frac{x^2 + \mu_n^2}{e^{\mu_n^2} + \mu_n^2} \right) \mu_n = 0 \]
\[ \sum_{n=0}^{\infty} \frac{\mu_n^2}{e^{\mu_n^2} + \mu_n^2} \mu_n = 0 \]
\[ \sum_{n=0}^{\infty} \frac{\mu_n^2}{e^{\mu_n^2} + \mu_n^2} \mu_n = 0 \]

**EXPAND THE SOURCE IN TERMS OF THE EIGENFUNCTIONS:**

\[ e^{-t} \sin x = \sum_{n=0}^{\infty} S_n(t) \sin(2n+1)x \quad \Rightarrow \quad S_n(t) = s_n e^{-t} \]

**NOW LET \( V(x,t) = \sum_{n=0}^{\infty} V_n(t) \sin \mu_n x \)**

\[ V_t = \sum_{n=0}^{\infty} V_n(t) \sin \mu_n x \]
\[ V_x = \sum_{n=0}^{\infty} \frac{\mu_n}{e^{\mu_n^2} + \mu_n^2} V_n(t) \sin \mu_n x \]

\[ \Rightarrow \quad V_t = V_{xx} - V + e^{-t} \sin x \]

**SINCE \( V \) SATISFIES HOMOGENEOUS BC ASSOCIATED WITH THE EIGENVALUES EIGENFUNCTIONS**

\[ \sum_{n=0}^{\infty} \left( \frac{\mu_n^2}{e^{\mu_n^2} + \mu_n^2} \right) \mu_n = 0 \]
\[ \sum_{n=0}^{\infty} \frac{\mu_n^2}{e^{\mu_n^2} + \mu_n^2} \mu_n = 0 \]
\[ \sum_{n=0}^{\infty} \frac{\mu_n^2}{e^{\mu_n^2} + \mu_n^2} \mu_n = 0 \]

**NOW LET \( V(x,t) = \sum_{n=0}^{\infty} V_n(t) \sin \mu_n x \)**

\[ V_n(t) = \sum_{n=0}^{\infty} \frac{\mu_n^2}{e^{\mu_n^2} + \mu_n^2} \mu_n = 0 \]
\[ \sum_{n=0}^{\infty} \frac{\mu_n^2}{e^{\mu_n^2} + \mu_n^2} \mu_n = 0 \]

**NOW \( 0 = \sum_{n=0}^{\infty} V_n(0) \sin \mu_n x \)**

\[ \Rightarrow \quad V_n(0) = C_n = 0. \]

\[ \Rightarrow \quad u(x,t) = \sum_{n=0}^{\infty} \left( \frac{e^{-t} - e^{-(1+\mu_n^2)^2}}{\mu_n^2} \right) s_n e^{-t} \sin \mu_n x \]

\[ = xe^{-t} + \sum_{n=0}^{\infty} \left( \frac{e^{-t} - e^{-(1+\mu_n^2)^2}}{\mu_n^2} \right) s_n \sin \mu_n x \]

\[ \mu_0 = 1. \]