1. Consider the differential equation

$$Ly = 8x^2y'' + 2xy' + (1+2x)y = 0 \qquad (1)$$

(a) Classify the points $0 \le x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

(a) $0 < x < \infty$ ARE ALL ORDINARY POINTS & $x=0$ IS A SINGULAR POINT.

$\lim_{x\to 0} x \frac{(2x)}{8x^2} = \frac{1}{4} = p_0 < \infty \quad \lim_{x\to 0} x^2 \frac{(1+2x)}{8x^2} = \frac{1}{8} = q_0 < \infty \Rightarrow x=0$ IS A REGULAR S.P.

(b) CONSIDER $L_0 y = x^2y'' + p_0 x y' + q_0 y = x^2 y'' + \frac{1}{4}xy' + \frac{1}{8}y = 0$

NOW LET $y = x^r \Rightarrow 8r(r-1) + 2r + 1 = 8r^2 - 6r + 1 = (2r-1)(4r-1) = 0 \Rightarrow r = \frac{1}{2}, \frac{1}{4}$.

(c) $Ly = 8x^2y'' + 2xy' + y + 2xy$

$$= \sum_{n=0}^{\infty} 8a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 2a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+1} = 0$$

CHANGE INDICES $m=n \quad m=n \quad m=n \quad m=n+1 \; n=m-1$

$$= \sum_{m=0}^{\infty} \{8(m+r)(m+r-1) + 2(m+r) + 1\} a_m x^{m+r} + \sum_{m=1}^{\infty} 2a_{m-1} x^{m+r} = 0 \qquad n=0 \Rightarrow m=1$$

$$= \{8r(r-1) + 2r + 1\} a_0 x^r + \sum_{m=1}^{\infty} \left[\{8(m+r)(m+r-1) + 2(m+r) + 1\} a_m + 2a_{m-1}\right] x^{m+r} = 0$$

EQUATE COEFFICIENTS TO 0:

$x^r]\quad 8r(r-1) + 2r + 1 = 8r^2 - 6r + 1 = (2r-1)(4r-1) = 0 \quad r = \frac{1}{2}, \frac{1}{4}$ (AS WE FOUND ABOVE)

$x^{m+r}{}_{m\ge 1}]\quad \left[\{8(m+r-1) + 2\}(m+r) + 1\right] a_m + 2a_{m-1} = 0$

OR $\quad a_m = \dfrac{-2a_{m-1}}{\{8(m+r) - 6\}(m+r) + 1} \quad$ THE RECURSION FOR $\{a_m\}$

$r = \frac{1}{2}$: $\quad a_m = \dfrac{-2a_{m-1}}{(8m+4-6)(m+\frac{1}{2}) + 1} = \dfrac{-2a_{m-1}}{(4m-1)(2m+1) - 1} = \dfrac{-2a_{m-1}}{8m^2 + 2m \cancel{-1} \cancel{+1}} = \dfrac{-a_{m-1}}{m(4m+1)}$

$a_1 = -\dfrac{a_0}{1.5} \quad a_2 = \dfrac{-a_1}{2.9} = \dfrac{+a_0}{90}, \ldots$

$r = \frac{1}{4}$: $\quad a_m = \dfrac{-2a_{m-1}}{(8m+2-6)(m+\frac{1}{4}) + 1} = \dfrac{-2a_{m-1}}{(2m-1)(4m+1) + 1} = \dfrac{-2a_{m-1}}{8m^2 - 2m \cancel{-1} \cancel{+1}} = \dfrac{-a_{m-1}}{m(4m-1)}$

$a_1 = -\dfrac{a_0}{1.3} \quad a_2 = \dfrac{-a_1}{2.7} = \dfrac{+a_0}{42}$

THUS

$$y(x) = a_0^{(1)} x^{1/2}\left[1 - \frac{x}{5} + \frac{x^2}{90} - \cdots\right] + a_0^{(2)} x^{1/4}\left[1 - \frac{x}{3} + \frac{x^2}{42} - \cdots\right]$$

2. Consider the following initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= u_{xx} - u\,, \qquad 0<x<1\,, \qquad t>0 \\ u_x(0,t) &= 0, \qquad u_x(1,t)=0 \\ u(x,0) &= x \end{aligned} \tag{2}$$

(a) Determine the solution to the boundary value problem (2) by separation of variables.

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(0,t) = 1$ with $O(\Delta x^2)$ accuracy.

Hint: Taylor's expansion may prove useful: $f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + O(\Delta x^3)$.

[6 marks]

[total 20 marks]

(a) LET $u(x,t) = X(x)T(t) \Rightarrow X\dot{T} = X''T - XT \Rightarrow \frac{\dot{T}}{T} + 1 = \frac{X''}{X} = -\lambda^2$

$T]$ $\dot{T} = -(1+\lambda^2)T \Rightarrow T(t) = Ce^{-(1+\lambda^2)t}$

$X]$ $\underline{\lambda \neq 0}$: $\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(1) \end{array}\right\}$ $\begin{array}{l} X = A\cos\lambda x + B\sin\lambda x \\ X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x \end{array}$

$X'(0) = B\lambda = 0 \Rightarrow B = 0 \quad X'(1) = -A\lambda\sin\lambda = 0 \Rightarrow \lambda_n = n\pi \quad n=1,2\ldots$

$X_n = \cos(n\pi x)$

$\underline{\lambda = 0}$: $X'' = 0 \quad X' = A \quad X = Ax + B \quad X'(0) = A = 0 \Rightarrow A = 0 \Rightarrow \lambda_0 = 0$

$X_0 = 1$

$$u(x,t) = A_0 e^{-t} + \sum_{n=1}^{\infty} A_n e^{-(1+\lambda_n^2)t}\cos(\lambda_n x)$$

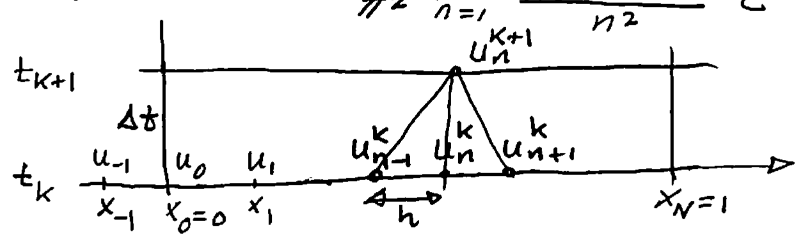
$$x = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos(n\pi x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x)$$

$$\therefore A_0 = \frac{a_0}{2} = \frac{2}{2}\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2},$$

$$A_n = a_n = \frac{2}{1}\int_0^1 x\cos(n\pi x)\,dx = \frac{2x\sin(n\pi x)}{(n\pi)}\Big|_0^1 - \frac{2}{(n\pi)}\int_0^1 \sin(n\pi x)\,dx = \frac{2}{(n\pi)^2}[\cos(n\pi)-1]$$

$$\therefore u(x,t) = e^{-t}/2 + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{[\cos(n\pi)-1]}{n^2}e^{-(1+(n\pi)^2)t}\cos(n\pi x)$$

(b)



$$u_{n\pm1}^k = u(x_n \pm h, t_k) = u_n^k \pm h\,u_n^{k\prime} + \frac{h^2}{2}u_n^{k\prime\prime} \pm \frac{h^3}{3!}u_n^{k(3)} + \frac{h^4}{4!}u_n^{k(4)} + \cdots \quad (*)$$

$$\therefore u_{n+1}^k + u_{n-1}^k = 2u_n^k + h^2 u_n^{k\prime\prime} + \frac{h^4}{12}u_n^{k(4)} \Rightarrow (u_{n+1}^k - 2u_n^k + u_{n-1}^k)/h^2 = \frac{\partial^2 u_n^k}{\partial x^2} + O(h^2)$$

AT $x = \cancel{h}$ INTRODUCE A GHOST POINT x_{-1} FROM $(*)$ WITH $n=0$ IT FOLLOWS THAT $\frac{\partial u_0^k}{\partial x} = \frac{u_1 - u_{-1}}{2h} + O(h^2)$

$$u_n^{k+1} = u_n^k + \Delta t\,\dot{u}_n^k + \frac{\Delta t^2}{2}\ddot{u}_n^k + \cdots \Rightarrow \frac{\partial u_n}{\partial t} = \frac{u_n^{k+1} - u_n^k}{\Delta t} + O(\Delta t)$$

$$\therefore u_n^{k+1} = u_n^k + (\Delta t/h^2)(u_{n+1}^k - 2u_n^k + u_{n-1}^k) + O(\Delta t, h^2)$$

3. The motion of a string subject to a gravitational load satisfies the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - g,\ 0 < x < 1,\ t > 0 \qquad (3)\\ u(0,t) &= u(1,t) = 0 \qquad (4)\\ u(x,0) &= \sin(\pi x),\ u_t(x,0) = 0. \end{aligned}$$

Here g is the acceleration due to gravity, which you may assume is constant.

(a) Determine the static deflection of the string, which is determined by solving (3) in which it is assumed that $u_{tt} = 0$ subject to the boundary conditions (4). [8 marks]

(b) Use the solution obtained in (a) to reduce the initial-boundary value problem to solving a homogeneous wave equation subject to homogeneous boundary conditions. Now use separation of variables to determine the solution to this boundary value problem and hence the complete solution of the complete initial-boundary value problem. [12 marks]

HINT: The following integral may be useful:

$$\int_0^1 (x^2 - x)\sin(n\pi x)\,dx = 2\frac{\cos n\pi - 1}{n^3\pi^3}$$

[total 20 marks]

(a) $0 = c^2 u^s_{xx} - g \Rightarrow u^s_{xx} = \frac{g}{c^2}$; $u^s_x = \frac{g}{c^2}x + A$, $u^s = \frac{g}{2c^2}x^2 + Ax + B$

$0 = u^s(0) = B$; $u^s(1) = \frac{g}{2c^2} + A = 0 \quad A = -g/2c^2$

$\therefore \boxed{u^s(x) = \frac{g}{2c^2}(x^2 - x)}$

(b) LET $u(x,t) = u^s(x) + v(x,t)$

$\therefore u_{tt} = \cancel{u^s_{tt}} + v_{tt} = c^2(u^s_{xx} + v_{xx}) - g = c^2 v_{xx} + [c^2 \cancel{u^s_{xx}} - g] \Rightarrow$

BC $\begin{cases} 0 = u^s(0) + v(0,t) = 0 + v(0,t) \\ 0 = u^s(1) + v(1,t) = 0 + v(1,t) \end{cases}$

IC: $\sin\pi x = u(x,0) = u^s(x) + v(x,0)$

$0 = u_t(x,0) = \cancel{u^s_t(x)} + v_t(x,0)$

$$\boxed{\begin{aligned} &v_{tt} = c^2 v_{xx}\\ \Rightarrow\ &v(0,t) = 0\\ \Rightarrow\ &v(1,t) = 0\\ \Rightarrow\ &v(x,0) = \sin(\pi x) - \tfrac{g}{2c^2}(x^2 - x)\\ \Rightarrow\ &v_t(x,0) = 0 \end{aligned}}$$

LET $v(x,t) = X(x)T(t) \Rightarrow \frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda^2$

X] $\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\}$ $\lambda_n = \frac{n\pi}{1}\ n = 1, 2, ..\ X_n = \sin(n\pi x)$

T] $\ddot{T} + \lambda^2 c^2 T = 0 \qquad T = e^{rt} \Rightarrow T = A\cos\lambda ct + B\sin\lambda ct$

$\therefore v(x,t) = \sum_{n=1}^{\infty}[A_n\cos\lambda_n ct + B_n\sin\lambda_n ct]\sin\lambda_n x$

$v_t(x,t) = \sum_{n=1}^{\infty}[-A_n\lambda_n c\sin\lambda_n ct + B_n\lambda_n c\cos\lambda_n ct]\sin\lambda_n x.$

$\sin(\pi x) - \frac{g}{2c^2}(x^2 - x) = v(x,0) = \sum_{n=1}^{\infty} A_n\sin\lambda_n x \Rightarrow A_n = \frac{2}{1}\int_0^1\{\sin(\pi x) - \frac{g}{2c^2}(x^2 - x)\}\sin n\pi x\,dx$

$\therefore A_n = \delta_{n1} - \frac{g}{c^2}\int_0^1 (x^2 - x)\sin(n\pi x)\,dx = \delta_{n1} - \frac{2g}{c^2}\left[\frac{\cos n\pi - 1}{n^3\pi^3}\right]$

$0 = v_t(x,0) = \sum_{n=1}^{\infty}(B_n\lambda_n c \cdot 1)\sin\lambda_n x \Rightarrow B_n = 0$

$\therefore u(x,t) = \frac{g}{2c^2}(x^2 - x) + \frac{2g}{c^2}\sum_{n=1}^{\infty}\frac{[1 - \cos(n\pi)]}{n^3\pi^3}\cos(n\pi ct)\sin(n\pi x) + \cos(\pi ct)\sin(\pi x)$

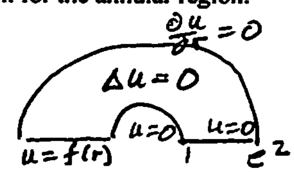
4. (a) Consider the eigenvalue problem

$$r^2R'' + rR' + \lambda R = 0 \qquad (1)$$
$$R(1) = 0 = R'(e^2) \qquad (2)$$

Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for the annular region:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < e^2, \quad 0 < \theta < \pi$$
$$u(r,0) = 0 \text{ and } u(r,\pi) = f(r)$$
$$u(1,\theta) = 0 \text{ and } \frac{\partial}{\partial r}u(e^2,\theta) = 0$$



[12 marks]
[total 20 marks]

(a) $-\frac{1}{r}\cdot(1) \Rightarrow -(rR''+R') = -(rR')' = \lambda\frac{1}{r}R \qquad R(1)=0=R'(e^2)$

WHICH IS A S-L PROBLEM WITH $p=r$ $q=0$ & $w=1/r$

LET $\lambda=\mu^2$ & LET $R=r^\gamma \Rightarrow \gamma(\gamma-1)+\gamma+\mu^2=\gamma^2+\mu^2=0 \quad \gamma=\pm i\mu \quad R=r^{\pm i\mu}=e^{i\mu\ln r}$

$\therefore R = A\cos\mu\ln r + B\sin\mu\ln r$

$R' = -A\sin\mu(\ln r)\cdot\frac{\mu}{r} + B\cos(\mu\ln r)\cdot(\mu/r)$

$0=R(1) = A\cos\mu(\ln 1)+B\sin(\mu\ln 1) = A \Rightarrow A=0$

$0=R'(e^2) = B\cos\mu(\ln e^2)\cdot\mu/e^2 = \frac{B\mu}{e^2}\cos 2\mu = 0$

NOW $\mu\neq 0$ OR WE HAVE THE TRIVIAL SOL'N SO THAT $\cos 2\mu = 0$

THUS $\mu_n = \frac{(2n+1)\pi}{4}, \ n=0,1,\dots \qquad R_n = \sin\mu_n\ln r$

(b) LET $u(r,\theta)=R(r)\Theta(\theta) \Rightarrow \frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\Theta} = -\mu^2$ (WANT AN EIGENVALUE PROBLEM IN $R(r)$)

$\boxed{R}$ $\left.\begin{array}{l} r^2R''+r^2R'+\mu^2R=0 \\ R(1)=0 \quad R'(e^2)=0 \end{array}\right\}$ $\mu_n=\frac{(2n+1)\pi}{4} \qquad R_n=\sin\mu_n\ln r$

$\boxed{\Theta}$ $\left.\begin{array}{l} \Theta''-\mu^2\Theta=0 \\ \Theta(0)=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \Theta(\theta)=A\cosh\mu\theta+B\sinh\mu\theta \\ \Theta(0)=A=0 \end{array}\right\} \Rightarrow \Theta_n(\theta)=\sinh(\mu_n\theta)$

$\therefore u(r,\theta)=\sum_{n=0}^{\infty} A_n\sinh(\mu_n\theta)\sin(\mu_n\ln r)$

$f(r)=u(r,\pi)=\sum_{n=0}^{\infty} A_n\sinh(\mu_n\pi)\sin(\mu_n\ln r)=\sum_{n=0}^{\infty} a_n^f\sin(\mu_n\ln r)$

$\therefore A_n\sinh(\mu_n\pi) = a_n^f = \int_1^{e^2}\frac{1}{r}f(r)\sin(\mu_n\ln r)\,dr \Big/ \int_1^{e^2}\frac{1}{r}\sin^2(\mu_n\ln r)\,dr$

NOW $\int_1^{e^2}\frac{1}{r}\sin^2(\mu_n\ln r)\,dr \overset{x=\ln r \quad dx=dr/r}{=} \int_0^2\sin^2(\mu_n x) = \frac{1}{2}\int_0^2 1-\cos(2\mu_n x)\,dx = \frac{1}{2}\left\{2-\sin\left(\frac{(2n+1)\pi}{2}x\right)\Big]_0^2\right\} = 1$

$\therefore A_n = \frac{1}{\sinh(\mu_n\pi)}\int_1^{e^2}\frac{1}{r}f(r)\sin(\mu_n\ln r)\,dr$

AND $u(r,\theta)=\sum_{n=0}^{\infty} A_n\sinh(\mu_n\theta)\sin(\mu_n\ln r)$

5. **Solve the inhomogeneous heat conduction problem:**

$$\begin{aligned} u_t &= u_{xx} + xt,\ 0 < x < 1,\ t > 0 \\ u_x(0,t) &= \frac{t^2}{2},\ \text{and } u(1,t) = 0 \\ u(x,0) &= 0. \end{aligned}$$

[20 marks]

LOOK FOR A FUNCTION $\omega(x,t) = A(t)x + B(t)$ THAT SATISFIES THE BC: $\omega_x = A(t) \Rightarrow \omega_x(0) = A(t) = t^2/2$

$$\omega(1,t) = \frac{t^2}{2}\cdot 1 + B = 0 \Rightarrow B = -t^2/2.$$

$\therefore \quad \omega(x,t) = t^2/2\,(x-1)$ SATISFIES THE BC.

NOW LET $u(x,t) = \omega(x,t) + V(x,t)$

$\therefore\ u_t = \omega_t + V_t = t(x-1) + V_t = \cancel{\omega_{xx}} + V_{xx} + xt \Rightarrow$

BC: $\cancel{t^2/2} = u_x(0,t) = \omega_x(0,t) + V_x(0,t) = \cancel{t^2/2} + V_x(0,t) \Rightarrow$

$0 = u(1,t) = \omega(1,t) + V(1,t) = 0 + V(1,t) \Rightarrow$

IC: $0 = u(x,0) = 0.(x-1) + V(x,0) \Rightarrow$

$$\boxed{\begin{aligned} V_t &= V_{xx} + t. \\ V_x(0,t) &= 0 \\ V(1,t) &= 0 \\ V(x,0) &= 0 \end{aligned}} \quad (2)$$

NOW USE AN EIGENFUNCTION EXPANSION TO SOLVE THE BVP (2). THE EIGENFUNCTIONS ASSOCIATED WITH THE BVP ARE THE SOLUTION TO $X'' + \lambda^2 X = 0,\ X'(0) = 0 = X(1)$

NAMELY $\lambda_n = \frac{(2n+1)\pi}{2}$ $n = 0,1,2,\ldots$ $X_n = \cos\lambda_n x$.

EXPAND THE SOURCE TERM IN TERMS OF THESE EIGENFUNCTIONS:

$$t.1 = \sum_{n=0}^{\infty} \hat{S}_n(t)\cos\lambda_n x \Rightarrow \hat{S}_n(t) = 2t\int_0^1 1.\cos\lambda_n x\,dx = 2t\,\frac{\sin\lambda_n x}{\lambda_n}\Big|_0^1 = 2t\frac{\sin\lambda_n}{\lambda_n}$$

NOW ASSUME $V(x,t) = \sum_{n=0}^{\infty} \hat{V}_n(t)\cos\lambda_n x \Rightarrow V_t = \sum_{n=0}^{\infty} \frac{d\hat{V}_n}{dt}\cos\lambda_n x,\ V_{xx} = \sum_{n=0}^{\infty} \hat{V}_n(-\lambda_n^2)\cos\lambda_n x$

$$\therefore\ V_t - V_{xx} - t = \sum_{n=0}^{\infty}\left\{\frac{d\hat{V}_n}{dt} + \lambda_n^2\hat{V}_n - 2t\frac{\sin\lambda_n}{\lambda_n}\right\}\cos\lambda_n x = 0$$

SINCE THE $\cos\lambda_n x$ ARE LINEARLY INDEPENDENT FUNCTIONS IT FOLLOWS THAT $\{\ \} = 0\ \forall n$.

$$\therefore\ \frac{d\hat{V}_n}{dt} + \lambda_n^2\hat{V}_n = 2t\frac{\sin\lambda_n}{\lambda_n} \Rightarrow \frac{d}{dt}\left\{e^{\lambda_n^2 t}\hat{V}_n\right\} = \frac{2\sin\lambda_n}{\lambda_n}\,t e^{\lambda_n^2 t}$$

$$\therefore\ e^{\lambda_n^2 t}\hat{V}_n = \frac{2\sin\lambda_n}{\lambda_n}\int_0^t s e^{\lambda_n^2 s}ds + C_n = \frac{2\sin\lambda_n}{\lambda_n}\left[\frac{s e^{\lambda_n^2 s}}{\lambda_n^2}\Big|_0^t - \int_0^t 1.\frac{e^{\lambda_n^2 s}}{\lambda_n^2}ds\right] + C_n$$

$$\therefore\ e^{\lambda_n^2 t}\hat{V}_n = \left[\frac{2\sin\lambda_n}{\lambda_n^3}\,t e^{\lambda_n^2 t} - \frac{2\sin\lambda_n}{\lambda_n^5}e^{\lambda_n^2 s}\Big|_0^t\right] + C_n$$

$$\therefore\ \hat{V}_n = \frac{2\sin\lambda_n}{\lambda_n^3}t - \frac{2\sin\lambda_n}{\lambda_n^5} + \frac{2\sin\lambda_n}{\lambda_n^5}e^{-\lambda_n^2 t} + C_n$$

$$\therefore\ V(x,t) = \sum_{n=0}^{\infty}\left\{\frac{2\sin\lambda_n}{\lambda_n^3}\left[t - \frac{(1-e^{-\lambda_n^2 t})}{\lambda_n^2}\right] + C_n\right\}\cos\lambda_n x$$

$$0 = V(x,0) = \sum_{n=0}^{\infty} C_n\cos\lambda_n x \Rightarrow C_n = 0$$

$$\therefore\ u(x,t) = \frac{t^2}{2}(x-1) + \sum_{n=0}^{\infty}\frac{2\sin\lambda_n}{\lambda_n^3}\left[t - \frac{(1-e^{-\lambda_n^2 t})}{\lambda_n^2}\right]\cos\lambda_n x.$$