SECTION 202 MATH 256/316

Q1 $u_t = u_{xx} - x e^{-t}$

$u_x(0,t) = (1 - e^{-t}) \qquad u(1,t) = 0$

$u(x,0) = 0$

LET $w(x,t) = a(t) + b(t)x \quad w_x = b(t) \quad w_x(0,t) = b(t) = (1-e^{-t})$

$w(1,t) = a(t) + (1-e^{-t}).1 \Rightarrow a(t) = -(1-e^{-t})$

$\therefore w(x,t) = (1-e^{-t})(x-1), \quad w_t(x,t) = +e^{-t}(x-1), \quad w_{xx} = 0$

LET $u(x,t) = w(x,t) + v(x,t)$

$0 = u_t - u_{xx} + x e^{-t} = w_t + v_t - \cancel{w_{xx}} - v_{xx} + x e^{-t} = v_t - v_{xx} + e^{-t}(x-1) + x e^{-t}$

$\boxed{v_t = v_{xx} + (1-2x)e^{-t}}$

$\cancel{1} e^{-t} = u_x(0,t) = w_x(0,t) + v_x(0,t) = \cancel{1 e^{-t}} + v_x(0,t) \Rightarrow \boxed{v_x(0,t) = 0}$

$0 = u(1,t) = w(1,t) + v(1,t) = 0 + v(1,t) \Rightarrow \boxed{v(1,t) = 0}$

$0 = u(x,0) = w(x,0) + v(x,0) = 0.(x-1) + v(x,0) \Rightarrow \boxed{v(x,0) = 0}$

SINCE THE EIGENFUNCTIONS FOR THE HOMOGENEOUS BVP ARE

$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X(1) \end{array}\right\} \quad \lambda_n = \frac{(2n+1)\pi}{2} \quad n = 0,1,2.. \quad X_n(x) = \cos\lambda_n x$

EXPAND THE SOURCE IN EIGENFUNCTIONS

$(1-2x)e^{-t} = S(x,t) = \sum_{n=0}^{\infty} \hat{S}_n(t) \cos\lambda_n x \Rightarrow \hat{S}_n(t) = \frac{2}{1}\int_0^1 (1-2x)e^{-t}\cos\lambda_n x\, dx$

$\therefore \hat{S}_n = 2e^{-t}\left\{ \frac{\sin\lambda_n x}{\lambda_n}\Big|_0^1 - 2\left[\frac{x \sin\lambda_n x}{\lambda_n}\Big|_0^1 + \frac{1}{\lambda_n}\int_0^1 \sin\lambda_n x\, dx \right] \right\} = 2e^{-t}\left\{ \frac{\sin\lambda_n}{\lambda_n} - \frac{2\sin\lambda_n}{\lambda_n} - \frac{2\cos\lambda_n x}{\lambda_n^2}\Big|_0^1 \right\}$

$= 2e^{-t}\left\{ -\frac{\sin\lambda_n}{\lambda_n} + \frac{2(1-\cos\lambda_n)}{\lambda_n^2} \right\}$

NOW LET

$v(x,t) = \sum_{n=0}^{\infty} \hat{v}_n(t) \cos\lambda_n x \Rightarrow 0 = v_t - v_{xx} - S(x,t) = \sum_{n=0}^{\infty} \left\{ \hat{v}_n' + \lambda_n^2 \hat{v}_n - \hat{S}_n(t) \right\} \cos(\lambda_n x)$

$\therefore \hat{v}_n' + \lambda_n^2 \hat{v}_n = 2e^{-t}\left\{ -\frac{\sin\lambda_n}{\lambda_n} + \frac{(1-\cos\lambda_n)}{\lambda_n^2} \right\} \Rightarrow \frac{d}{dt}\left[e^{\lambda_n^2 t} \hat{v} \right] = 2e^{(\lambda_n^2-1)t}\left\{ -\frac{\sin\lambda_n}{\lambda_n} + \frac{2(1-\cos\lambda_n)}{\lambda_n^2} \right\}$

$\therefore e^{\lambda_n^2 t} \hat{v}_n(t) = \frac{2e^{(\lambda_n^2-1)t}}{(\lambda_n^2-1)}\left\{ -\frac{\sin\lambda_n}{\lambda_n} + 2\frac{(1-\cos\lambda_n)}{\lambda_n^2} \right\} + c_n$

$\therefore \hat{v}_n(t) = \frac{2e^{-t}}{(\lambda_n^2-1)}\left\{ -\frac{\sin\lambda_n}{\lambda_n} + \frac{2(1-\cos\lambda_n)}{\lambda_n^2} \right\} + c_n . e^{-\lambda_n^2 t}$

$\therefore v(x,t) = \sum_{n=0}^{\infty} \left\{ \frac{2e^{-t}}{\lambda_n^2-1}\left[-\frac{\sin\lambda_n}{\lambda_n} + \frac{2(1-\cos\lambda_n)}{\lambda_n^2} \right] + c_n e^{-\lambda_n^2 t} \right\} \cos\lambda_n x$

$0 = v(x,0) = \sum_{n=0}^{\infty} \left\{ \frac{2}{\lambda_n^2-1}\left[-\frac{\sin\lambda_n}{\lambda_n} + \frac{2(1-\cos\lambda_n)}{\lambda_n^2} \right] + c_n \right\} \cos\lambda_n x.$

$\therefore c_n = \frac{-2}{\lambda_n^2-1}\left[-\frac{\sin\lambda_n}{\lambda_n} + \frac{2(1-\cos\lambda_n)}{\lambda_n^2} \right]$

$\therefore u(x,t) = (1-e^{-t})(x-1) + \sum_{n=0}^{\infty} \frac{2}{(\lambda_n^2-1)}\left[-\frac{\sin\lambda_n}{\lambda_n} + \frac{2(1-\cos\lambda_n)}{\lambda_n^2} \right]\left(e^{-t} - e^{-\lambda_n^2 t} \right)\cos\lambda_n x$

Q2 $\quad u_{tt} = c^2 u_{xx}$

$u(0,t) = 0 \quad u_x(1,t) = 0$

$u(x,0) = 1 \quad u_t(x,0) = 0.$

EIGENFUNCTIONS ARE $X_n = \sin(\lambda_n x)$ & EIGENVALUES $\lambda_n = \frac{(2n+1)\pi}{2}$ $n = 0, 1, 2 \ldots$

LET $u(x,t) = \sum_{n=0}^{\infty} \hat{u}_n(t) \sin(\lambda_n x)$

$$0 = u_{tt} - c^2 u_{xx} = \sum_{n=0}^{\infty} \left\{ \frac{d^2 \hat{u}_n}{dt^2} + \lambda_n^2 c^2 \hat{u}_n \right\} \sin(\lambda_n x)$$

$$\therefore \frac{d^2 \hat{u}_n}{dt^2} + \lambda_n^2 c^2 \hat{u}_n = 0 \Rightarrow \hat{u}_n(t) = A_n \cos(\lambda_n c t) + B_n \sin \lambda_n(ct)$$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} \left\{ A_n \cos(\lambda_n c t) + B_n \sin(\lambda_n c t) \right\} \sin(\lambda_n x).$$

$$1 = u(x,0) = \sum_{n=0}^{\infty} A_n \sin(\lambda_n x) \Rightarrow A_n = \frac{2}{1} \int_0^1 1.\sin \lambda_n x \, dx = \frac{2(1-\cos\lambda_n)}{\lambda_n}$$

$$u_t(x,t) = \sum_{n=0}^{\infty} \left\{ -A_n(\lambda_n c) \sin(\lambda_n c t) + B_n(\lambda_n c) \cos(\lambda_n c t) \right\} \sin(\lambda_n x)$$

$$0 = u_t(x,0) = \sum_{n=0}^{\infty} B_n \lambda_n c \sin(\lambda_n x) \Rightarrow B_n = 0$$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} \overbrace{\left\{ \frac{2(1-\cos\lambda_n)}{\lambda_n} \right\}}^{A_n} \sin(\lambda_n x) \cos(\lambda_n c t)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} A_n \left\{ \sin \lambda_n (x+ct) + \sin \lambda_n (x-ct) \right\}$$

$$= \frac{1}{2} \left\{ F(x+ct) + F(x-ct) \right\}$$ SIMILAR TO D'ALEMBERT'S SOL'N

WHERE

$$F(x) = \sum_{n=0}^{\infty} \frac{2(1-\cos\lambda_n)}{\lambda_n} \sin(\lambda_n x) = \sum_{n=0}^{\infty} \frac{2\left[1-\cos\left(\frac{(2n+1)\pi}{2}\right)\right]}{\left(\frac{(2n+1)\pi}{2}\right)} \sin\left(\frac{(2n+1)\pi x}{2}\right)$$

WAVELENGTH $\sin\left(\frac{(2n+1)\pi}{2}(x+\Delta)\right) = \sin\left(\frac{(2n+1)\pi x}{2}\right)$

$$\Delta \frac{(2n+1)\pi}{2} = 2\pi \Rightarrow \Delta_n = \frac{4}{(2n+1)}$$

$\Delta_0 = 4$

PERIOD = 4