1. $u_t = u_{xx} + 1 \qquad 0 < x < \frac{\pi}{2} \quad t > 0$

$u(0,t) = t \quad u_x(\frac{\pi}{2}, t) = e^{-\gamma t^2} \quad 0 < \gamma < 1$

$u(x,0) = x$

CONSTRUCT $W(x,t) = A(t)x + B(t)$ TO MATCH THE NONZERO BC

$W(0,t) = B(t) = t \qquad W_x = A(t) \Rightarrow W_x(\frac{\pi}{2}, t) = A(t) = e^{-\gamma t} \Rightarrow W(x,t) = e^{-\gamma t}x + t$

NOW LET $u(x,t) = W(x,t) + V(x,t)$

$u_t = (W_t + V_t) = (-\gamma e^{-\gamma t}x + \cancel{1} + V_t) = (\cancel{W_{xx}} + V_{xx}) + \cancel{1} \Rightarrow V_t = V_{xx} + \gamma e^{-\gamma t}x$

$\cancel{t} = u(0,t) = W(0,t) + V(0,t) = \cancel{t} + V(0,t) \qquad \Rightarrow V(0,t) = 0$

$\cancel{e^{-\gamma t}} = u_x(\frac{\pi}{2}, t) = W_x(\frac{\pi}{2}, t) + V_x(\frac{\pi}{2}, t) = \cancel{e^{-\gamma t}} + V_x(\frac{\pi}{2}, t) \qquad \Rightarrow V_x(\frac{\pi}{2}, t) = 0$

$\cancel{x} = u(x,0) = W(x,0) + V(x,0) = \cancel{x} + V(x,0) \qquad \Rightarrow V(x,0) = 0$

$\therefore$ THE EIGENFUNTIONS & EIGENVALUES ASSOCIATE WITH THE MIXED BC

$\lambda_n = \frac{(2n+1)\pi}{2(\pi/2)} = (2n+1) \qquad X_n = \sin(2n+1)x$

LET $\gamma e^{-\gamma t}x = \sum_{n=0}^{\infty} S_n(t) \sin \lambda_n x \Rightarrow S_n(t) = \frac{2\gamma}{(\pi/2)} \int_0^{\pi/2} e^{-\gamma t} x \sin(2n+1)x\,dx$

$\therefore S_n(t) = \frac{4\gamma}{\pi} e^{-\gamma t} \left\{ -x \frac{\cos(2n+1)x}{(2n+1)} \Big|_0^{\pi/2} + \frac{1}{(2n+1)} \int_0^{\pi/2} 1 \cos(2n+1)x\,dx \right\} = \frac{4\gamma}{\pi} e^{-\gamma t} \frac{\sin(2n+1)x}{(2n+1)^2} \Big|_0^{\pi/2} = \frac{4(-1)^n \gamma}{\pi(2n+1)^2} e^{-\gamma t} = \sigma_n e^{-\gamma t}$

LET $V(x,t) = \sum_{n=0}^{\infty} V_n(t) \sin \lambda_n x \qquad V_t = \sum_{n=0}^{\infty} \dot{V}_n(t) \sin \lambda_n x \qquad V_{xx} = \sum_{n=0}^{\infty} V_n(t) \{-\lambda_n^2\} \sin \lambda_n x$

$\therefore V_t - V_{xx} - \gamma e^{-\gamma t} x = \sum_{n=0}^{\infty} \{ \dot{V}_n + \lambda_n^2 V_n - \sigma_n e^{-\gamma t} \} \sin \lambda_n x = 0 \Rightarrow \{\ \} = 0$ FOR EACH n.

$\therefore \frac{d}{dt}\{e^{\lambda_n^2 t} V_n\} = e^{\lambda_n^2 t} \dot{V}_n + e^{\lambda_n^2 t} \lambda_n^2 V_n = \sigma_n e^{(\lambda_n^2 - \gamma)t}$

$e^{\lambda_n^2 t} V_n = \sigma_n \int_0^t e^{(\lambda_n^2 - \gamma)\tau} d\tau + C_n = \frac{\sigma_n}{\lambda_n^2 - \gamma} [e^{(\lambda_n^2 - \gamma)t} - 1] + C_n$

$V_n(t) = \frac{\sigma_n}{\lambda_n^2 - \gamma} (e^{-\gamma t} - e^{-\lambda_n^2 t}) + C_n e^{-\lambda_n^2 t}$

$0 = V(x,0) = \sum_{n=0}^{\infty} V_n(0) \sin \lambda_n x = \sum_{n=0} \{ \frac{\sigma_n}{\lambda_n^2 - \gamma} (\cancel{1 - 1}) + C_n \} \sin \lambda_n x \Rightarrow C_n = 0$

$\therefore u(x,t) = e^{-\gamma t} x + t + \frac{4\gamma}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n^2 - \gamma} (e^{-\gamma t} - e^{-(2n+1)^2 t}) \sin(2n+1)x$

2. $u_{tt} = u_{xx} + \gamma \sin x \quad + (x/\pi) \qquad 0<x<\pi \quad t>0$

$u(0,t)=0 \qquad u(\pi,t)=t^2/2$

$u(x,0)=0 \qquad u_t(x,0)=\sin(3x)$

a) LET $W(x,t)=A(t)x+B(t) \qquad W(0,t)=B(t)=0$; $W(\pi,t)=A(t)\pi = t^2/2 \quad A(t)=t^2/2\pi$

$\therefore W(x,t)=\frac{t^2}{2}\left(\frac{x}{\pi}\right)$

b) LET $u(x,t)=\frac{t^2x}{2\pi}+V(x,t)$

$u_{tt} = \left(\frac{t^2x}{2\pi}+V\right)_{tt} = \cancel{\frac{x}{\pi}}+V_{tt} = \left(\cancel{\frac{t^2x}{2\pi}}+V\right)_{xx}+\gamma\sin x+\cancel{(x/\pi)} \Rightarrow V_{tt}=V_{xx}+\gamma\sin x$

$0=u(0,t)=W(0,t)+V(0,t)=0+V(0,t) \qquad \Rightarrow V(0,t)=0$

$\cancel{\frac{t^2}{2}}=u(\pi,t)=W(\pi,t)+V(\pi,t)=\cancel{\frac{t^2}{2}}+V(\pi,t) \qquad \Rightarrow V(\pi,t)=0$

$0=u(x,0)=W(0,t)+V(x,0)=0+V(x,0) \qquad \Rightarrow V(x,0)=0$

$\sin(3x)=u_t(x,0)=W_t(x,0)+V_t(x,0)=\cancel{\frac{tx}{\pi}}\Big|_{t=0}+V_t(x,0) \qquad \Rightarrow V_t(x,0)=\sin(3x)$

(c)

M1) THE EIGENVALUES & EIGENFUNCTIONS ASSOCIATED WITH THE HOMOGENEOUS BC ARE

$\lambda_n=\frac{n\pi}{\pi}=n \quad n=1,2\ldots \qquad X_n=\sin(nx)$

LET $\gamma\sin x=\sum_{n=1}^{\infty} s_n\sin(nx) \qquad s_n=\gamma\delta_{n1}$

$V(x,t)=\sum_{n=1}^{\infty} V_n(t)\sin(nx) \quad V_{tt}=\sum_{n=1}^{\infty} \ddot V_n \sin(nx) \quad V_{xx}=\sum_{n=1}^{\infty} V_n(-n^2)\sin(nx)$

$V_{tt}-V_{xx}-\gamma\sin x = \sum_{n=1}^{\infty}\{\ddot V_n+n^2V_n-\delta_{n1}\gamma\}\sin nx=0 \Rightarrow \{\ \}=0$

$\ddot V_n+n^2V_n=\delta_{n1}\gamma \qquad V_n=a_n\cos nt+b_n\sin nt + \delta_{n1}\gamma/n^2$ PARTICULAR SOL'N

$0=V(x,0)=\sum_{n=1}^{\infty} V_n(0)\sin nx=\sum_{n=1}^{\infty}\{a_n+\delta_{n1}\gamma/n^2\}\sin nx \Rightarrow a_n=-\delta_{n1}\gamma/n^2.$

$V_t(x,t)=\sum_{n=1}^{\infty} \dot V_n(t)\sin nx = \sum_{n=1}^{\infty}\{-a_n n\sin nt+b_n n\cos nt+0\}\sin(nx)$

$\sin 3x=V_t(x,0)=\sum_{n=1}^{\infty} b_n n\sin(nx) \Rightarrow b_n=\delta_{n3}/n$

$\therefore u(x,t)=\frac{t^2}{2}\left(\frac{x}{\pi}\right)+\sum_{n=1}^{\infty}\{(1-\cos t)\delta_{n1}\gamma/n^2+\delta_{n3}/n \sin nt\}\sin nx$

$=\frac{t^2}{2}\left(\frac{x}{\pi}\right)-\gamma\cos t\sin x+\frac13\sin3t\sin(3x)+\gamma\sin x$

$=(t^2x/2\pi)-\frac{\gamma}{2}\{\sin(x+t)+\sin(x-t)\}+\frac16\{\cos3(x-t)-\cos3(x+t)\}+\gamma\sin x$

M2) LOOK FOR A STEADY SOL'N $\Omega(x)$ OF THE V EQ: $\Omega_{tt}=0=\Omega_{xx}+\gamma\sin x$

$\therefore \Omega_{xx}=-\gamma\sin x \quad \Omega_x=\gamma\cos x+A \therefore \Omega=\gamma\sin x+Ax+B \quad \Omega(0)=0\Rightarrow B=0 \quad \Omega(\pi)=B=0.$

$\therefore$ LET $V(x,t)=\Omega(x)+\psi(x,t) \Rightarrow V_{tt}=(\cancel{\Omega}+\psi)_{tt}=\psi_{xx}+\{\cancel{\Omega_{xx}+\gamma\sin x}\} \Rightarrow \psi_{tt}=\psi_{xx}$

$0=V(0,x)=\Omega(0)+\psi(0,t)=0+\psi(0,t)\Rightarrow\psi(0,t)=0$; $0=V(\pi,t)=\Omega(\pi)+\psi(\pi,t)=0+\psi(\pi,t)\Rightarrow\psi(\pi,t)=0$

$0=V(x,0)=\Omega(x)+\psi(x,0)\Rightarrow\psi(x,0)=-\gamma\sin x$; $\sin3x=V_t(x,0)=\cancel{\Omega(x)}_t+\psi_t(x,0)\Rightarrow\psi_t(x,0)=\sin3x$

SEPARATING VARIABLES $\psi(x,t)=\sum_{n=1}^{\infty}\{A_n\cos nt+B_n\sin(nt)\}\sin(nx)$

$-\gamma\sin x=\psi(x,0)=\sum_{n=1}^{\infty} A_n\sin(nx) \quad A_n=-\gamma\delta_{n1}$; $\sin3x=\psi_t(x,0)=\sum_{n=1}^{\infty} B_n n\cos nt\Big|_{t=0}\sin(nx)\Rightarrow B_n=\frac{\delta_{n3}}{n}$

$\therefore u(x,t)=(t^2x/2\pi)+\gamma\sin x+\sum_{n=1}^{\infty}\{(-\gamma\delta_{n1})\cos nt+(\delta_{n3}/n)\sin nt\}\sin nx$

$=(t^2x/2\pi)-\gamma\cos t\sin x+\frac13\sin3t\sin3x-\gamma\sin x$ AS BEFORE

(d) BONUS QUESTION: ASSUME $\gamma = 0$ $v(x,t)$ SATISFIES

$$v_{tt} = v_{xx}$$

$$\text{BC: } v(0,t) = 0 = v(\pi,t)$$

$$\text{IC: } v(x,0) = f(x) = 0 \quad v_t(x,0) = g(x) = \sin(3x)$$

SINCE v IS SUBJECT TO HOMOGENEOUS DIRICHLET BC

$$v(x,t) = \frac{1}{2}\left[f_0(x-t) + f_0(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g_0(s)\,ds$$

WHERE $f_0(x)$ IS THE ODD EXTENSION OF PERIOD 2π OF f AND

$g_0(x)$ " " " " " " " " g

$$\therefore \; v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \sin(3s)\,ds = -\frac{1}{6}\cos 3s\Big|_{x-t}^{x+t} = \frac{1}{6}\{\cos 3(x-t) - \cos 3(x+t)\}$$

$\therefore$ $u(x,t) = t^2 x/2\pi + \frac{1}{6}\{\cos 3(x-t) - \cos 3(x+t)\}$ AS ABOVE WITH $\gamma = 0$.