MATH 257/316 MIDTERM 2 SECTION 102

1. $u_t = u_{xx} + x \qquad 0 < x < \pi/2,\ t > 0$

BC: $u(0,t) = e^{-\gamma t} \quad u_x(\pi/2, t) = t$

IC: $u(x,0) = 1 + \sin 3x$

FIND A FUNCTION $W(x,t) = A(t)x + B(t)$ THAT SATISFIES THE NONZERO BC

$e^{-\gamma t} = W(0,t) = B(t) \qquad t = W_x(\pi/2, t) = A(t) \Rightarrow W(x,t) = tx + e^{-\gamma t}$

LET $u(x,t) = W(x,t) + V(x,t)$ THEN DETERMINE THE PDE, BC & IC SATISFIED BY $V(x,t)$.

$u_t = (W_t + V_t) = \cancel{x} - \gamma e^{-\gamma t} + V_t = (\cancel{W_{xx}} + V_{xx}) + \cancel{x} \Rightarrow V_t = V_{xx} + \gamma e^{-\gamma t}$

$\cancel{e^{-\gamma t}} = u(0,t) = W(0,t) + V(0,t) = \cancel{e^{-\gamma t}} + V(0,t) \Rightarrow V(0,t) = 0$

$\cancel{t} = u_x(\pi/2, t) = W_x(\pi/2, t) + V_x(\pi/2, t) = \cancel{t} + V_x(\pi/2, t) \Rightarrow V_x(\pi/2, t) = 0$

$\cancel{1} + \sin 3x = u(x,0) = W(x,0) + V(x,0) = \cancel{1} + V(x,0) \Rightarrow V(x,0) = \sin 3x.$

BY SEPARATING VARIABLES THE EIGENVALUES & EIGENFUNCTIONS ASSOCIATED WITH THE HOMOGENEOUS BC ON V ARE $\lambda_n = (2n+1)\pi/2(\pi/2) = (2n+1),\ n = 0, 1, \dots$ AND $X_n(x) = \sin(2n+1)x$

EXPAND $S(x,t) = \gamma e^{-\gamma t}$ AND $V(x,t)$ IN TERMS OF THESE EIGENFUNCTIONS

$S(x,t) = \gamma e^{-\gamma t} = \sum_{n=0}^{\infty} \hat{S}_n(t) \sin(\lambda_n x)\ ;\ \hat{S}_n = \frac{4\gamma}{\pi}\int_0^{\pi/2} e^{-\gamma t} \sin\lambda_n x\, dx = \frac{4\gamma e^{-\gamma t}}{\pi}\left[\frac{-\cos\lambda_n x}{\lambda_n}\right]_0^{\pi/2} = \frac{e^{-\gamma t} 4\gamma}{\pi\lambda_n}\left[1 - \cos\frac{(2n+1)\pi}{2}\right] = c_n \gamma e^{-\gamma t}$

LET $V(x,t) = \sum_{n=0}^{\infty} \hat{V}_n(t) \sin\lambda_n x \quad V_t = \sum_{n=0}^{\infty} \frac{d\hat{V}_n}{dt} \sin\lambda_n x \quad V_{xx} = \sum_{n=0}^{\infty} \hat{V}_n(-\lambda_n^2)\sin(\lambda_n x)$

$0 = V_t - V_{xx} - \gamma e^{-\gamma t} = \sum_{n=0}^{\infty}\left\{\frac{d\hat{V}_n}{dt} + \lambda_n^2 \hat{V}_n - \gamma e^{-\gamma t} c_n\right\}\sin\lambda_n x$ SINCE $\sin\lambda_n x$ ARE LIN. INDEPENDENT $\{\ \} = 0$

$\therefore \quad \frac{d\hat{V}_n}{dt} + \lambda_n^2 \hat{V}_n = \gamma e^{-\gamma t} c_n \Rightarrow \frac{d}{dt}\left[e^{\lambda_n^2 t}\hat{V}_n\right] = e^{(\lambda_n^2 - \gamma)t}\gamma c_n$

$\therefore \quad e^{\lambda_n^2 t}\hat{V}_n = \gamma c_n e^{+(\lambda_n^2 - \gamma)t}/(\lambda_n^2 - \gamma) + d_n$

$\therefore \quad \hat{V}_n(t) = \frac{\gamma c_n e^{-\gamma t}}{(\lambda_n^2 - \gamma)} + d_n e^{-\lambda_n^2 t}$

$\therefore \quad V(x,t) = \sum_{n=0}^{\infty}\left[\frac{\gamma c_n e^{-\gamma t}}{(\lambda_n^2 - \gamma)} + d_n e^{-\lambda_n^2 t}\right]\sin\lambda_n x.$

NOW $\sin 3x = V(x,0) = \sum_{n=0}^{\infty}\left[\frac{\gamma c_n}{\lambda_n^2 - \gamma} + d_n\right]\sin(2n+1)x$

MATCHING COEFFICIENTS $d_n = \begin{cases} -\gamma c_n/(\lambda_n^2 - \gamma) & n \neq 1 \\ -\gamma c_n/(\lambda_n^2 - \gamma) + 1 & n = 1 \end{cases}$

$$\therefore\ u(x,t) = xt + e^{-\gamma t} + e^{-9t}\sin 3x + \gamma\sum_{n=0}^{\infty} c_n\left[\frac{e^{-\gamma t} - e^{-\lambda_n^2 t}}{\lambda_n^2 - \gamma}\right]\sin(2n+1)x.$$

$$= xt + e^{-\gamma t} + e^{-9t}\sin 3x + \frac{4\gamma}{\pi}\sum_{n=0}^{\infty}\frac{e^{-\gamma t} - e^{-(2n+1)^2 t}}{(2n+1)^2 - \gamma}\sin(2n+1)x.$$

2. $u_{tt} = u_{xx} - 2 \qquad 0 < x < 1,\ t > 0$

$u(0,t) = 0 = u(1,t)$

$u(x,0) = x^2 - x \qquad u_t(x,0) = \sin 2\pi x.$

a) $0 = W_{xx} - 2 \qquad W_x = 2x + A \qquad W = x^2 + Ax + B$

$0 = W(0) = B \qquad 0 = W(1) = 1 + A.1 \Rightarrow A = -1 \qquad W(x) = x^2 - x.$

b) $u(x,t) = W(x) + V(x,t)$ STEADY SOL'N

PDE: $U_{tt} = (\cancel{W_{tt}} + V_{tt}) = (W_{xx} + V_{xx}) - 2 = \{\cancel{W_{xx} - 2}\} + V_{xx} \Rightarrow V_{tt} = V_{xx}$

BC: $0 = u(0,t) = W(0) + V(0,t) = 0 + V(0,t) \Rightarrow V(0,t) = 0$

$0 = u(1,t) = W(1) + V(1,t) = 0 + V(1,t) \Rightarrow V(1,t) = 0$

IC: $\cancel{x^2 - x} = u(x,0) = W(x) + V(x,0) = \cancel{x^2 - x} + V(x,0) \Rightarrow V(x,0) = 0$

$\sin 2\pi x = u_t(x,0) = \cancel{W_t} + V_t(x,0) = V_t(x,0) \Rightarrow V_t(x,0) = \sin 2\pi x.$

c) LET $V(x,t) = X(x)\,T(t) \Rightarrow \ddot{T}/T = X''/X = -\lambda^2$ CONSTANT

$X'' + \lambda^2 X = 0$ & $X(0) = 0 = X(1) \Rightarrow \lambda_n = n\pi \quad n = 1, 2, \ldots, \quad X_n = \sin(n\pi x)$

$\ddot{T} + \lambda_n^2 T = 0, \quad T(t) = A\cos\lambda_n t + B\sin(\lambda_n t)$

$V(x,t) = \sum_{n=1}^{\infty} [A_n \cos\lambda_n t + B_n \sin\lambda_n t]\sin\lambda_n x$

$V_t(x,t) = \sum_{n=1}^{\infty} [-A_n\lambda_n \sin\lambda_n t + B_n\lambda_n \cos\lambda_n t]\sin\lambda_n x$

$0 = V(x,0) = \sum_{n=1}^{\infty} A_n \sin\lambda_n x \Rightarrow A_n = 0$

$\sin(2\pi x) = \sum_{n=1}^{\infty} B_n\lambda_n \sin(n\pi x) \Rightarrow B_n = 0 \quad n \neq 2 \qquad B_2 = 1/2\pi$

$\therefore V(x,t) = \frac{1}{2\pi}\sin(2\pi t)\sin 2\pi x = \frac{1}{4\pi}[\cos 2\pi(x-t) - \cos 2\pi(x+t)]$

$\therefore u(x,t) = x^2 - x - \frac{1}{4\pi}[\cos 2\pi(x+t) - \cos 2\pi(x-t)]$

d) $V(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \sin(2\pi s)\,ds = -\frac{1}{4\pi}[\cos 2\pi(x+t) - \cos 2\pi(x-t)]$ AS ABOVE

$\therefore u(x,t) = x^2 - x - \frac{1}{4\pi}[\cos 2\pi(x+t) - \cos 2\pi(x-t)]$