MATH 257/316 MIDTERM 2 SECTION 102 OCT. 2012. SOLUTIONS

Q1. $u_t = u_{xx} - x + e^{-t}\sin(3\pi x) \quad 0<x<1,\ t>0$

$u(0,t)=t \quad u(1,t)=0$

$u(x,0)=0.$

SOL'N: LOOK FOR A SIMPLE FUNCTION $\omega(x,t)$ THAT IS CONSTRUCTED TO SATISFY THE BC.

LET $\omega(x,t) = A(t)x + B(t)$, $\omega(0,t) = B(t) = t$, $\omega(1,t) = t + A(t).1 = 0 \quad A(t) = -t.$

$\therefore \boxed{\omega(x,t) = t(1-x)}$

NOW LET $u(x,t) = \omega(x,t) + v(x,t)$

PDE: $u_t = \omega_t + v_t = (1-x) + v_t = \cancel{\omega_{xx}} + v_{xx} - x + e^{-t}\sin(3\pi x) \Rightarrow v_t = v_{xx} - 1 + e^{-t}\sin(3\pi x)$

BC: $t = u(0,t) = \omega(0,t) + v(0,t) = t + v(0,t) \Rightarrow v(0,t)=0$

$0 = u(1,t) = \omega(1,t) + v(1,t) = 0 + v(1,t) \Rightarrow v(1,t) = 0$

IC: $0 = u(x,0) = \omega(x,0) + v(x,0) = 0 + v(x,0) \Rightarrow v(x,0) = 0$

v SATISFIES A PDE WITH HOMOGENEOUS BC FOR WHICH THE EIGENVALUES ARE $\lambda_n = n\pi \quad n=1,2,\ldots$ AND EIGENFUNCTIONS ARE $X_n = \sin(n\pi x)$

EXPAND THE SOURCE FUNCTION $-1+e^{-t}\sin(3\pi x)$ AS A SERIES OF EIGENFUNCTIONS

$S(x,t) = -1 + e^{-t}\sin(3\pi x) = \sum_{n=1}^{\infty}(b_n + e^{-t}\delta_{n3})\sin n\pi x$ WHERE $\delta_{n3} = \begin{cases}1 & n=3\\ 0 & n\neq 3\end{cases}$

AND $b_n = -\frac{2}{1}\int_0^1 \sin(n\pi x) = +2\left[\frac{\cos n\pi x}{n\pi}\right]_0^1 = \frac{2}{n\pi}[\cos(n\pi)-1] = \begin{cases}\frac{-4}{n\pi} & n \text{ ODD}\\ 0 & n \text{ EVEN}\end{cases}$

NOW LET $v(x,t) = \sum_{n=1}^{\infty}\hat{v}_n(t)\sin\lambda_n x$; $v_t = \sum_{n=1}^{\infty}\dot{\hat{v}}_n \sin(\lambda_n x)$; $v_{xx} = \sum_{n=1}^{\infty}\hat{v}_n(-\lambda_n^2)\sin\lambda_n x$

$\therefore v_t - v_{xx} + 1 - e^{-t}\sin(3\pi x) = \sum_{n=1}^{\infty}\left\{\frac{d\hat{v}_n}{dt} + \lambda_n^2\hat{v}_n - b_n - \delta_{n3}e^{-t}\right\}\sin(n\pi x) = 0.$

SINCE $\sin(n\pi x)$ ARE L.I $\{\}=0 \Rightarrow e^{\lambda_n^2 t}\dot{\hat{v}}_n + e^{\lambda_n^2 t}\lambda_n^2\hat{v}_n = \frac{d}{dt}\left[e^{\lambda_n^2 t}\hat{v}_n\right] = e^{\lambda_n^2 t}b_n + \delta_{n3}e^{(\lambda_n^2-1)t}$

$\therefore \quad e^{\lambda_n^2 t}\hat{v}_n = \frac{e^{\lambda_n^2 t}}{\lambda_n^2}b_n + \delta_{n3}\frac{e^{(\lambda_n^2-1)t}}{\lambda_n^2-1} + c_n$

$\therefore \quad \hat{v}_n(t) = b_n/\lambda_n^2 + \delta_{n3}\frac{e^{-t}}{\lambda_n^2-1} + c_n e^{-\lambda_n^2 t}$

$\therefore \quad v(x,t) = \sum_{n=1}^{\infty}\left\{\frac{b_n}{\lambda_n^2} + \frac{\delta_{n3}e^{-t}}{\lambda_n^2-1} + c_n e^{-\lambda_n^2 t}\right\}\sin\lambda_n x$

$0 = v(x,0) = \sum_{n=1}^{\infty}\left\{\frac{b_n}{\lambda_n^2} + \frac{\delta_{n3}}{\lambda_n^2-1} + c_n\right\}\sin\lambda_n x \Rightarrow c_n = -\frac{b_n}{\lambda_n^2} - \frac{\delta_{n3}}{\lambda_n^2-1}$

$\therefore \quad v(x,t) = \sum_{n=1}^{\infty}\frac{b_n}{\lambda_n^2}(1-e^{-\lambda_n^2 t})\sin\lambda_n x + \frac{e^{-t} - e^{-\lambda_3^2 t}}{(\lambda_3^2-1)}\sin(3\pi x)$

$\therefore u(x,t) = t(1-x) + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{[\cos(n\pi)-1]}{n^3}(1-e^{-(n\pi)^2 t})\sin(n\pi x) + \frac{e^{-t} - e^{-(3\pi)^2 t}}{(3\pi)^2-1}\sin(3\pi x)$

2. $u_{tt} = C^2 u_{xx}$

$u_x(0,t) = 0 \qquad u_x(1,t) = 0$

$u(x,0) = 1 - x \qquad u_t(x,0) = 0.$

ANS: SINCE WE HAVE HOMOGENEOUS BC SEPARATE VARIABLES. $u(x,t) = X(x)T(t)$

$$X\ddot{T} = c^2 X'' T \Rightarrow \frac{\ddot{T}}{c^2 T(t)} = \frac{X''}{X(x)} = -\lambda^2$$

$T]\quad \ddot{T} + \lambda^2 c^2 T = 0 \qquad T = A\cos\lambda ct + B\sin\lambda ct\ ;\ T = A + Bt$ WHEN $\lambda = 0$

$X]\quad \left.\begin{matrix} X'' + \lambda^2 X = 0 \\ \lambda \neq 0:\ X'(0) = 0 = X'(1) \end{matrix}\right\} \quad \lambda_n = n\pi \quad n = 1, 2, \ldots \quad X_n = \cos(n\pi x)$

$\left.\begin{matrix} \lambda = 0:\ X'' = 0 \\ X'(0) = 0 = X'(1) \end{matrix}\right\} \quad \lambda_0 = 0 \quad X_0 = 1.$

$$u(x,t) = (A_0 + B_0 t) \cdot 1 + \sum_{n=1}^{\infty} \left\{A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right\} \cos(\lambda_n x)$$

$$u_t(x,t) = B_0 + \sum_{n=1}^{\infty} \left\{-A_n \lambda_n c \sin(\lambda_n ct) + B_n \lambda_n c \cos(\lambda_n ct)\right\} \cos\lambda_n x$$

$$1 - x = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

$$A_0 = \frac{2}{2}\int_0^1 (1-x)\,dx = \frac{1}{2}, \qquad A_n = \frac{2}{1}\int_0^1 (1-x)\cos(n\pi x)\,dx = 2\,\frac{[1 - \cos(n\pi)]}{n^2\pi^2}$$

$$0 = u_t(x,0) = B_0 + \sum_{n=1}^{\infty} (B_n \lambda_n c)\cos(\lambda_n x) \Rightarrow B_0,\ B_n = 0$$

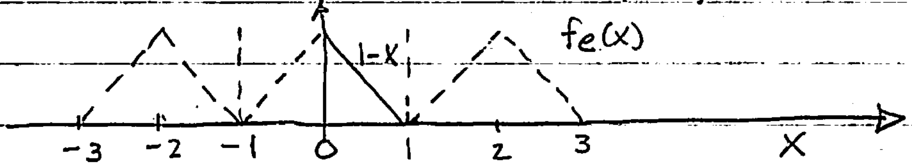
$$\therefore\ u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n ct)\cos(\lambda_n x)$$

BUT $\frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] = \cos\alpha\cos\beta$

$$\therefore\ u(x,t) = \frac{1}{2}\left[A_0 + \sum_{n=1}^{\infty} A_n \cos\lambda_n(x+ct)\right] + \frac{1}{2}\left[A_0 + \sum_{n=1}^{\infty} A_n \cos\lambda_n(x-ct)\right]$$

$= \frac{1}{2} f_e(x+ct) + \frac{1}{2} f_e(x-ct)$ WHICH CORRESPONDS TO D'ALEMBERT'S SOLUTION

WHERE f_e IS THE EVEN EXTENSION OF $f(x) = 1 - x$ SHOWN BELOW:



$$u(x,2) = \frac{1}{2}\left[A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi(x+2)\right] + \frac{1}{2}\left[A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi(x-2)\right]$$

$$= \frac{1}{2}\left[A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x\right] + \frac{1}{2}\left[A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\right] = f_e(x)$$

WHICH IS PLOTTED ABOVE.