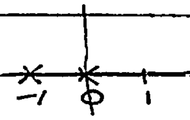
MATH 257/316 MIDTERM 1 SECTION 201 FEB 2012



1. $Ly = 16x^2(1+x)y'' + 16xy' - y = 0$

(a) IF $x \notin \{0,-1\}$ THEN x IS AN ORDINARY POINT

$x=0$ IS A S.P. $\lim_{x\to 0} \frac{x(16x)}{16x^2(1+x)} = 1 = p_0 < \infty$; $\lim_{x\to 0} \frac{x^2(-1)}{16x^2(1+x)} = -\frac{1}{16} = q_0 < \infty \Rightarrow x=0$ RSP

$x=-1$ IS A SP: $\lim_{x\to -1} \frac{\cancel{(x+1)}\,16x}{16x^2\cancel{(x+1)}} = -1 = p_0 < \infty$, $\lim_{x\to -1} \frac{(x+1)^2(-1)}{16x^2(x+1)} = 0 = q_0 < \infty \Rightarrow x=-1$ IS A RSP

(b) SINCE $x=1$ IS AN ORDINARY POINT WE AN EXPANSION $y(x) = \sum_{n=0}^{\infty} C_n(x-1)^n$

SINCE THE CLOSEST SINGULAR POINT TO $x_0=1$ IS $x=0$ WE EXPECT

THE MINIMAL RADIUS OF CONVERGENCE IS $\rho \geq |1-0| = 1$

(c) SINCE $x=1$ IS A RSP WE ASSUME A FROBENIUS EXPANSION OF THE FORM

$y(x) = \sum_{n=0}^{\infty} C_n (x+1)^{n+r}$ WHERE r IS A PARAMETER

- TO DETERMINE THE BEHAVIOUR OF $y(x)$ AS $x\to -1$ WE CONSIDER THE LOWEST ORDER OPERATOR $L_0 y = (x+1)^2 y'' + p_0 (x+1) + q_0 y = -(x+1)^2 y'' - (x+1)y' = 0$ (SEE (a) ABOVE)

MAKING THE SUBSTITUTION $y = x^r \Rightarrow r(r-1) - r = r^2 - 2r = r(r-2) = 0 \Rightarrow r = 0, 2$

THUS $y_1(x) = C_0 + C_1(x+1) + \cdots$ & $y_2(x) = (x+1)^2[d_0 + d_1(x+1) + \cdots]$

THUS THE ONE SOLUTION $y_1(x) \xrightarrow{x\to -1} C_0$ AND $y_2(x) \xrightarrow{x\to -1} d_0(x+1)^2$

- THE MINIMAL RADIUS OF CONVERGENCE IS $\rho \geq 1$ SINCE THE NEAREST SINGULAR POINT IS AT $x=0$ A DISTANCE 1 AWAY FROM -1.

(d) THE INDICIAL EQUATION IS $r(r-1) + r - \frac{1}{16} = r^2 - \frac{1}{16} = 0 \quad r = \pm\frac{1}{4}$

LET $y(x) = \sum_{n=0}^{\infty} C_n x^{n+r}$, $y'(x) = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 16x^2 y'' + 16x^3 y'' + 16xy' - y = 0$$

$$= \sum_{n=0}^{\infty} 16 C_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0} 16 C_n (n+r)(n+r-1) x^{n+r+1} + \sum_{n=0}^{\infty} 16 C_n (n+r) x^{(n+r)} - \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$\boxed{n=m}$ $\quad$ $n+1=m$ $\boxed{n=m-1}$ $n=0 \Rightarrow m=1$ $\quad$ $\boxed{n=m}$ $\quad$ $\boxed{n=m}$

$$= \sum_{m=0}^{\infty} [16(m+r)(m+r-1) + 16(m+r) - 1] C_m x^{m+r} + \sum_{m=1}^{\infty} 16 C_{m-1} (m+r-1)(m+r-2) x^{m+r}$$

$$= [16r(r-1) + 16r - 1] C_0 x^r + \sum_{m=1}^{\infty} \{C_m [16(m+r)(m+r-1+1) - 1] + 16(m+r-1)(m+r-2) C_{m-1}\} x^{m+r}$$

$x^r >$ $16r^2 - 1 = 0 \quad r = \pm 1/4$ INDICIAL EQ

$x^{m+r}, m \geq 1 >$ $C_m[16(m+r)^2 - 1] + 16(m+r-1)(m+r-2) C_{m-1} = 0$

RECURSION $C_m = -\dfrac{16(m+r-1)(m+r-2) C_{m-1}}{16(m+r)^2 - 1}$

$r = -1/4$: $\quad C_m = \dfrac{-16(m-5/4)(m-9/4)\,C_{m-1}}{16(m-\frac{1}{4})^2-1} = -\dfrac{(4m-5)(4m-9)\,C_{m-1}}{(4m-1)^2-1}$

$$= \frac{-(4m-5)(4m-9)\,C_{m-1}}{16m^2-8m+\cancel{1}-\cancel{1}} = -\frac{(4m-5)(4m-9)\,C_{m-1}}{8m(2m-1)}$$

$\therefore \quad C_1 \overset{m=1}{=} -\dfrac{(4-5)(4-9)}{8.1(2-1)}\,C_0 = -\dfrac{5}{8}\,C_0$

$C_2 \overset{m=2}{=} -\dfrac{(8-5)(8-9)}{16(4-1)}\,C_1 = +\dfrac{\cancel{3}.1}{16.\cancel{3}}\,C_1 = -\dfrac{5}{128}\,C_0$

$\therefore \; y_1(x) = C_0 X^{-1/4}\left[1 - \dfrac{5}{8}X - \dfrac{5}{128}X^2 + \cdots\right]$

$r = 1/4$: $\quad C_m = \dfrac{-16(m-3/4)(m-7/4)C_{m-1}}{16(m+1/4)^2-1} = \dfrac{-(4m-3)(4m-7)\,C_{m-1}}{(4m+1)^2-1}$

$$= \frac{-(4m-3)(4m-7)C_{m-1}}{16m^2+8m+1-1} = \frac{-(4m-3)(4m-7)}{8m(2m+1)}\,C_{m-1}$$

$C_1 \overset{m=1}{=} \dfrac{-(4-3)(4-7)}{8.(2+1)}\,C_0 = \dfrac{+3}{8.3}\,C_0 = \dfrac{C_0}{8}$

$C_2 \overset{m=2}{=} -\dfrac{(8-3)(8-7)}{16(5)}\,C_1 = \dfrac{-\cancel{5}.1}{16.\cancel{5}}\,C_1 = -\dfrac{C_0}{128}$

$\therefore \; y_2(x) = C_0 X^{1/4}\left[1 + \dfrac{X}{8} - \dfrac{X^2}{128} + \cdots\right]$

Q 2 $\quad u_t = u_{xx} \quad 0 < x < \pi \quad t > 0$

$u(0,t) = 0 = u_x(\pi, t)$

$u(x,0) = x(\pi - x)$

LET $\quad u(x,t) = X(x)\,T(t)$

$X(x)\dot{T}(t) = X''(x)\,T(t)$

$\div XT: \quad \dfrac{\dot{T}(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda^2$ CONST

TIME EQ: $\quad \dot{T}(t) = -\lambda^2 T(t) \Rightarrow T(t) = D e^{-\lambda^2 t}$.

SPACE EQ: $\lambda \neq 0$: $X'' + \lambda^2 X = 0$, $X(0) = 0 = X'(\pi)$ EIGENVALUE PROBLEM.

$X(x) = A\cos\lambda x + B\sin\lambda x$

$X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$0 = X(0) = A \Rightarrow A = 0$

$0 = X'(\pi) = B\lambda\cos(\lambda\pi) = 0 \Rightarrow \lambda_k \pi = \dfrac{(2k+1)\pi}{2} \quad k = 0, 1, 2 \ldots$

$\therefore \quad \lambda_k = (2k+1)/2 \quad k = 0, 1, 2 \ldots$

$\lambda = 0$: $\quad X'' = 0 \quad X = A + Bx \quad X(0) = A = 0 \quad X'(x) = B \Rightarrow X'(\pi) = B = 0$ TRIVIAL

$\therefore$ THE EIGENVALUES ARE $\lambda_k = (2k+1)/2 \quad k = 0, 1, 2 \ldots$

& EIGENFUNCTIONS ARE $X_k(x) = \sin\left(\dfrac{(2k+1)}{2}x\right)$.

THE SOLUTION OF THE HEAT EQ IS THUS OF THE FORM

$$u(x,t) = \sum_{k=0}^{\infty} B_k \sin\left(\frac{(2k+1)x}{2}\right) e^{-\left(\frac{2k+1}{2}\right)^2 t}$$

NOW $\quad x(\pi - x) = u(x,0) = \displaystyle\sum_{k=0}^{\infty} B_k \sin\left(\frac{(2k+1)}{2}x\right)$

WHERE $\quad B_k = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x(\pi - x)\sin\left(\frac{(2k+1)x}{2}\right)dx = \dfrac{8}{\pi}\,\dfrac{4 + (-1)^{k+1}(2k+1)\pi}{(2k+1)^3}$

$$\therefore \quad u(x,t) = \frac{8}{\pi}\sum_{k=0}^{\infty}\left\{\frac{4 + (-1)^{k+1}(2k+1)\pi}{(2k+1)^3}\right\}\sin\left(\frac{(2k+1)x}{2}\right) e^{-\left(\frac{2k+1}{2}\right)^2 t}.$$