MATH 257/316 MIDTERM 1 SECTION 102 OCT 2012

1(a) • $X=0$ IS THE ONLY S.P. WHILE ALL OTHER POINTS $0<X<\infty$ ARE ORDINARY POINTS

• CLASSIFICATION OF $X=0$:

$$\lim_{X\to 0} X\frac{\{-(1-X)X\}}{3X^2} = -\frac{1}{3} = p_0 < \infty \quad \lim_{X\to 0} X^2\frac{1}{3X^2} = \frac{1}{3} = q_0 < \infty \Rightarrow X=0 \text{ IS A REGULAR SP.}$$

INDICIAL EQ: $L_0 y = X^2 y'' - \frac{X}{3}y' + \frac{1}{3}y = 0 \quad y = x^r \Rightarrow r(r-1) - \frac{1}{3}r + \frac{1}{3} = r^2 - \frac{4}{3}r + \frac{1}{3} = (r-1)(r-\frac{1}{3}) = 0 \Rightarrow r = 1, \frac{1}{3}$

(b) SINCE $X=1$ IS AN ORDINARY POINT WE ASSUME $y(x) = \sum_{n=0}^{\infty} a_n (X-1)^n$

SINCE THE CLOSEST SINGULAR POINT IS AT $X=0$ THE MINIMAL RADIUS OF CONVERGENCE IS $\rho = 1$.

(c) SINCE $X=0$ IS A RSP WE ASSUME $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 3x^2 y'' \qquad - xy' \qquad + x^2 y' \qquad + y$$

$$= \sum_{n=0}^{\infty} 3a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$m=n \qquad m=n \qquad m=n+1 \quad n=m-1 \qquad m=n$

$n=0 \Rightarrow m=1$

$$= \sum_{m=0}^{\infty} [3a_m (m+r)(m+r-1) - a_m (m+r) + a_m] x^{m+r} + \sum_{m=1}^{\infty} a_{m-1} (m+r-1) x^{m+r}$$

$$= \{3r(r-1) - r + 1\} a_0 x^r + \sum_{m=1}^{\infty} \{[3(m+r)(m+r-1) - (m+r) + 1] a_m + (m+r-1) a_{m-1}\} x^{m+r}$$

$x^r]$ $\quad a_0\{3r^2 - 4r + 1\} = a_0(3r-1)(r-1) = 0 \quad r = 1, \frac{1}{3}$

x^{m+r} $m \geq 1]$ $\quad [(m+r)\{3(m+r) - 4\} + 1] a_m = -(m+r-1) a_{m-1}$

$r=1$: $$a_m = \frac{-m\, a_{m-1}}{(m+1)(3m-1)+1} = \frac{-\cancel{m}\, a_{m-1}}{\cancel{m}(3m+2)}$$

$$a_1 = -a_0/5 \qquad a_2 = -a_1/8 = a_0/40 \qquad a_3 = -a_2/11 = -a_0/440$$

$$\therefore\ y(x) = a_0^{(1)} x^1 [1 - x/5 + x^2/40 - x^3/440 + \cdots]$$

$r=\frac{1}{3}$ $$a_m = \frac{-(m-2/3)\, a_{m-1}}{(m+\frac{1}{3})\{3m+1-4\}+1} = \frac{-(m-2/3)\, a_{m-1}}{(3m+1)(m-1)+1} = \frac{-\frac{1}{3}(\cancel{3m-2})\, a_{m-1}}{(\cancel{3m-2})m} = \frac{-a_{m-1}}{3m}$$

$$a_1 = \frac{-a_0}{3} \qquad a_2 = \frac{-a_1}{6} = \frac{a_0}{18} \qquad a_3 = \frac{-a_2}{9} = \frac{-a_0}{162}$$

$$\therefore\ y(x) = a_0^{(2)} x^{1/3} \left[1 - \frac{x}{3} + \frac{x^2}{18} - \frac{x^3}{162} + \cdots\right]$$

2. $u_t = u_{xx} \qquad 0 < x < \pi \qquad t > 0$

$u_x(0,t) = 0 = u_x(\pi, t)$

$u(x,0) = x$

LET $u(x,t) = X(x)\,T(t)$

$\Rightarrow \dot{T}(t) X(x) = X''(x) T(t)$

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

T] $\dot{T} = -\lambda^2 T \Rightarrow T(t) = D e^{-\lambda^2 t}$

X] $\lambda \neq 0$: $X'' + \lambda^2 X = 0$, $X(0) = 0 = X(\pi)$; $X(x) = A\cos\lambda x + B\sin\lambda x$, $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = B\lambda = 0 \overset{\lambda\neq 0}{\Rightarrow} B = 0 \qquad X'(\pi) = -A\lambda\sin\lambda\pi = 0 \Rightarrow \lambda\pi = n\pi,\ n = 1,2\ldots$

$\therefore \lambda_n = n \quad n = 1,2\ldots \quad X_n(x) = \cos(nx)$

$\lambda = 0$: $X'' = 0 \Rightarrow X = Ax + B \quad X' = A \quad X'(0) = A = 0 \Rightarrow X = B.1$

$\therefore \lambda_0 = 0 \quad X_0(x) = 1.$

THE SOLUTION IS OF THE FORM

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos(nx)$$

$$x = u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left\{ \frac{x\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} 1.\sin(nx)\,dx \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{\cos(nx)}{n^2}\Big|_0^{\pi} \right\} = \frac{2}{\pi}\left\{\frac{(-1)^n - 1}{n^2}\right\} = \begin{cases} -4/\pi n^2 & n \text{ ODD} \\ 0 & n \text{ even} \end{cases}$$

$$\therefore u(x,t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\{(-1)^n - 1\}}{n^2} e^{-n^2 t}\cos(nx)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t}}{(2k+1)^2}\cos((2k+1)x)$$