1. \( Ly = 2x^2y'' + 3xy' + (2x-1)y' = 0 \)

a) \( 0 \leq x \leq 0 \) are ordinary points

\( x = 0 \) is a singular point \( \lim_{x \to 0} x(3x) = 3 \neq 0 \) and \( \lim_{x \to 0} x^2(2x-1) = 0 \), \( p(x) \) is not \( \infty \) at \( x = 0 \).

Consider \( L y = x^2y'' + 3xy' - y = 0 \). Given \( y = x^r \) \( r/2 = 1 \), \( r = 2 \), \( r - 1 = 1 \) or \( 2r^2 + r - 1 = 0 \), \( r = \frac{1}{2} \) and \( r = -1 \) are the roots of the indicial eq.

b) About \( x_0 = -1 \), which is an ordinary point, assume \( y(x) = \sum_{n=0}^{\infty} a_n(x + 1)^n \)

The distance from \( x = -1 \) to \( x = 0 \), the nearest singular point is \( 1 - 2 = 1 \) so \( p \geq 1 \).

c) About the singularity \( x = 0 \), assume \( y = \sum_{n=0}^{\infty} a_n X^{r+n} \), \( y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1} \), \( y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} \),

\[ 2y'' + 3xy' - y = 0 \]

\[ \sum_{n=0}^{\infty} \left( r + n \right) \left( r + n - 1 \right) a_n x^{r+n-2} + \sum_{n=0}^{\infty} 3a_n x^{r+n-1} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0 \]

\[ a_n \left[ 2(r-1) + 3r - 1 \right] - \frac{2a_n}{m+n+1} = 0 \]

\[ x^r \] \( 2r^2 + r - 1 = (2r - 1)(r + 1) = 0 \) \( r = \frac{1}{2}, -1 \) as before

\[ a_m = \frac{-2a_{m-1}}{(m+r)[(m+1)(m+2) - 1]} \]

\[ T = \frac{1}{2} \Rightarrow a_0 = -2a_{-1} = -2a_0 = \frac{2a_0}{5} \]

\[ a_1 = -2a_0 = -2a_0 \]

\[ a_1 = 2a_0 = \frac{2a_0}{35} \]

\[ H(x) = a_0 x^{1/2} \left[ 1 - 2x/5 + \frac{2x^2}{35} - \cdots \right] \]

\[ T = -1 \Rightarrow a_m = -2a_{m-1} = -2a_{m-1} = -2a_{m-1} \]

\[ a_0 = 2a_1 = 2a_0 = \frac{2a_0}{2(1)} \]

\[ H(x) = a_0 x^{-1} \left[ 1 + 2x - 2x^2 - \cdots \right] \]

Since there are no other singularities for \( x = 0 \), the radius of convergence is \( \infty \).
\[ u_t = u_{xx} + y^2 u, \quad 0 < x < \pi/2, \quad t > 0 \]

\[ u(0,t) = 0 = u_x(\pi/2, t) \]

\[ u(x,0) = \sin 3x. \]

**LET** \( u(x,t) = \tilde{u}(x) \exp(\lambda t) \Rightarrow u_t = \tilde{u}_t = u_{xx} + y^2 u = \tilde{u}'' + \lambda \tilde{u} \)

\[ \frac{\tilde{u}(0)}{\tilde{T}(\tilde{t})} - \frac{\tilde{u}'(0)}{\tilde{T}(\tilde{t})} = \frac{\tilde{u}(\pi)}{\tilde{T}(\tilde{t})}, \quad \tilde{T}(\tilde{t}) = C e^{(\lambda + y^2)\tilde{t}} \]

\[ \tilde{u}'' = \lambda \tilde{u}, \quad \tilde{u}(0) = 0 = \tilde{u}(\pi/2) \]

\[ \lambda > 0: \text{say} \quad \lambda = \mu^2 \Rightarrow \tilde{u}'' - \mu^2 \tilde{u} = 0 \]

\[ \tilde{u} = A \cos \mu x + B \sin \mu x \]

\[ \tilde{u}' = A \mu \sin \mu x + B \mu \cos \mu x \]

\[ 0 = \tilde{u}(0) = A \quad 0 = \tilde{u}'(\pi/2) = A \mu \sin \mu \pi/2 = B \Rightarrow \tilde{u} \equiv 0 \quad \text{TRIVIAL SOLN} \]

\[ \lambda = 0: \tilde{u}'' = 0 \quad \tilde{u}' = A \quad \tilde{u} = A x + B \]

\[ 0 = \tilde{u}(0) = B \quad 0 = \tilde{u}'(\pi/2) = A \Rightarrow \tilde{u} \equiv 0 \quad \text{THE TRIVIAL SOLN} \]

\[ \lambda < 0: \text{say} \quad \lambda = -\mu^2 \Rightarrow \tilde{u}'' + \mu^2 \tilde{u} = 0 \]

\[ \tilde{u} = A \cos \mu x + B \sin \mu x \]

\[ \tilde{u}' = -A \mu \sin \mu x + B \mu \cos \mu x \]

\[ 0 = \tilde{u}(0) = A \quad 0 = \tilde{u}'(\pi/2) = B \mu \cos \mu \pi/2 = -B \mu \Rightarrow \mu = (2n-1)\pi/2, \quad n = 1, 2, \ldots \]

\[ \lambda_n = -\mu_n^2 = (2n-1)^2 \pi^2/4, \quad n = 1, 2, \ldots \]

\[ \text{THE EIGENVALUES AND} \]

\[ \tilde{u}_n(x) = \sin((2n-1)x) \quad \text{THE CORRESPONDING EIGENFUNCTIONS} \]

\[ \text{Thus} \quad u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(2n-1)^2 t} e^{x^2 t} \sin((2n-1)x) \]

\[ \text{NOW} \quad \sin 3x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin((2n-1)x) \]

\[ \text{SINCE THE} \ \tilde{x}_n \ \text{ARE LINEARLY INDEPENDENT WE MAY EQUATE COEFFICIENTS} \]

\[ \text{TO OBTAIN} \quad b_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases} \]

\[ u(x,t) = e^{-9t} e^{x^2 t} \sin 3x \]