MATH 257/316 MIDTERM 1 SECTION 101 OCT 2013

1. $Ly = 3xy'' + y' - y = 0 \qquad P = 3x \quad Q = 1 \quad R = -1$

a) $x > 0$ ARE ALL ORDINARY POINTS, $x = 0$ IS A SINGULAR POINT

$x = 0$: $\lim_{x\to 0} x\left(\frac{1}{3x}\right) = \frac{1}{3} = p_0 \quad \lim_{x\to 0} x^2\left(\frac{-1}{3x}\right) = 0 = q_0 < \infty \Rightarrow x_0 = 0$ IS A REGULAR S.P.

THE INDICIAL EQ IS THUS $r(r-1) + \frac{1}{3}r = 0 \quad 3r^2 - 2r = r(3r-2) = 0 \Rightarrow r = 0, 2/3$

b) SINCE $x = -2$ IS AN ORDINARY POINT SO WE ASSUME A SERIES EXPANSION OF THE FORM $y(x) = \sum_{n=0}^{\infty} a_n (x+2)^n$

THE RADIUS OF CONVERGENCE IS AT LEAST AS LARGE AS THE DISTANCE BETWEEN $x_0 = -2$ AND THE NEAREST SINGULAR POINT $x = 0 \Rightarrow \rho \geq |-2 - 0| = 2$.

c) LET $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 3xy'' + y' - y$$

$$= \sum_{n=0}^{\infty} 3a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$m = n \qquad m = n \qquad m-1 = n \Rightarrow n = 0 \Rightarrow m = 1$

$$= \sum_{m=0}^{\infty} [3a_m (m+r)(m+r-1) + a_m (m+r)] x^{m+r-1} - \sum_{m=1}^{\infty} a_{m-1} x^{m+r-1}$$

$$= a_0 [3r(r-1) + r] x^{r-1} + \sum_{m=1}^{\infty} \{a_m [(m+r)\{3(m+r-1)+1\}] - a_{m-1}\} x^{m+r-1}$$

x^{r-1}) $r(3r-2) = 0 \quad r = 0, 2/3$

x^{m+r-1}) $$a_m = \frac{a_{m-1}}{(m+r)[3(m+r)-2]} \qquad m \geq 1$$

$r = 2/3$: $$a_m = \frac{a_{m-1}}{(m+2/3)[3m+\cancel{2}-\cancel{2}]} = \frac{a_{m-1}}{m(3m+2)}$$

$$a_1 = \frac{a_0}{1.5} \qquad a_2 = \frac{a_1}{2.8} = \frac{a_0}{80}$$

$$y_1(x) = a_0 x^{2/3} [1 + x/5 + x^2/80 + \cdots]$$

$r = 0$: $$a_m = \frac{a_{m-1}}{m(3m-2)}$$

$$a_1 = \frac{a_0}{1.1} \qquad a_2 = \frac{a_1}{2.4} = \frac{a_0}{8}$$

$$y_2(x) = a_0 x^0 \left[1 + x + \frac{x^2}{8} + \cdots\right]$$

2. $u_t = u_{xx}$ $\quad 0<x<\pi,\ t>0$ $\quad (1)$

BC: $u(0,t) = 0 = \dfrac{\partial u(\pi,t)}{\partial t}$ $\quad (2)$

IC: $u(x,0) = x.$ $\quad (3)$

LET $u(x,t) = X(x)T(t)$ AND SUB

$$\Rightarrow \quad \frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda^2$$

T> $\dot{T} = -\lambda^2 T \Rightarrow T(t) = Ce^{-\lambda^2 t}.$

X> $\lambda \neq 0$: $X'' + \lambda^2 X = 0$ $\quad X(0) = 0 = X'(\pi)$, $\quad X = e^{rx} \Rightarrow r^2 + \lambda^2 = 0$ $\quad r = \pm\lambda i$

$X = A\cos\lambda x + B\sin\lambda x$ $\quad X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$0 = X(0) = A\cdot 1 + B\cdot 0 \Rightarrow A = 0$

$0 = X'(\pi) = B\lambda\cos\lambda\pi = 0 \Rightarrow \lambda\pi = (2n+1)\pi/2$ $\quad n = 0,1,\ldots$

$\lambda_n = (2n+1)/2$ $\quad n = 0,1,2,\ldots$ $\quad X_n(x) = \sin[(2n+1)x/2]$

$\lambda = 0$: $X'' = 0$ $\quad X' = A$ $\quad X = Ax + B$ $\quad X(0) = B = 0$, $X'(\pi) = A = 0 \Rightarrow X = 0.$

$$\therefore \quad u(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)x}{2}\right) e^{-\left(\frac{2n+1}{2}\right)^2 t}$$

IC:

$$x = f(x) = u(x,0) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)x}{2}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin\left(\frac{(2n+1)x}{2}\right)dx = \frac{8}{\pi}\frac{(-1)^n}{(2n+1)^2}$$ FROM GIVEN INTEGRAL.

$$\therefore \quad u(x,t) = \frac{8}{\pi}\sum_{n=0}^{\infty} e^{-\left(\frac{2n+1}{2}\right)^2 t}\frac{(-1)^n}{(2n+1)^2}\sin\left(\frac{(2n+1)x}{2}\right)$$