Q1(a) \( (x^2-4)^2 y'' + 8(x+2) y' + y = 0 \)

\( x = \pm 2 \) are singular points.

\( x = 2 : \lim_{x \to 2} \frac{8(x+2)}{(x-2)^2(x+2)^2} \to \infty \)

So \( x = 2 \) is an irregular SD.

\( x = -2 : \lim_{x \to -2} \frac{8(x+2)}{(x+2)(x-2)^2} = \frac{8}{16} = \frac{1}{2} = \phi_0 \)

\( \lim_{x \to -2} \frac{(x+2)^2}{(x+2)(x-2)^2} = \frac{1}{16} = \rho_0 \)

Since \( 1\phi_0 \times 1\rho_0 \times = \infty \) \( x = -2 \) is a regular singular point.

Indicial Eq: \( \tau(T-1) + \tau + \frac{1}{16} = 0 \)

\( (4\tau-1)(4\tau+1) = 0 \) \( \tau = \frac{1}{4}, \frac{1}{4} \) is a double root.

\( y(x) = A x^{1/4} \left[ 1 + \ldots \right] + B x^{-1/4} \left[ 1 + \ldots \right] \)

Q2(b) \( (x-3)(x+1) y'' + (x-3) y' + (x+1) y = 0 \)

\( x = 3 \) and \( x = -1 \) are singular points.

\( x = -1 : \lim_{x \to -1} \frac{(x+1)}{(x-3)(x-3)} = 1 = \phi_0 \)

\( \lim_{x \to -1} \frac{(x+1)^2}{(x+1)(x-3)} = 0 = \rho_0 \)

Since \( 1\phi_0 \times 1\rho_0 \times = \infty \) \( x = -1 \) is a regular SD.

Indicial Eq: \( \tau(T-1) + \tau + 0 = 0 \) \( \tau = 0, 0 \) is a double root.

\( x = 3 : \lim_{x \to 3} \frac{(x-3)}{(x-3)(x+1)} = 0 = \phi_0 \)

\( \lim_{x \to 3} \frac{(x-3)^2 (x+1)}{(x+1)(x+1)} = 0 \)

\( T - (T-1) + 0 = \tau \tau = 0 = 0, 0 \) are the roots.
\[ L y = 2 x y'' + y' - x y = 0 \]

As \( x \to 0 \), \( \frac{x}{2x} = \frac{1}{2} = p_0 \)

As \( x \to 0 \), \( x^2 (-x) = 0 = q_0 \)

Is a regular singular point

\[ r (r-1) + \frac{1}{2} r = r^2 - \frac{1}{2} r = 0 \quad r = 0, \frac{1}{2} \]

\[ y = \sum_{n=0}^{\infty} a_n x^{n+r} \]

\[ y' = \sum_{n=0}^{\infty} n a_n (n+r) x^{n+r-1} \]

\[ y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \]

\[ \frac{m}{m+1} \]

\[ \sum_{n=0}^{\infty} 2 a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} = 0 \]

\[ \sum_{n=0}^{\infty} \frac{a_m (m+r) [2 (m+r) - 1]}{m!} x^{m+r} = 0 \]

\[ a_0 \left[ r (2r-2) \right] = 0 \quad r = 0, \frac{1}{2} \]

\[ a_1 (r+1)(2r+1) = 0 \quad r = 0 = a_1, 1, l = 0 \Rightarrow q_1 = 0 \]

\[ a_{1/2} (3/2)(3/2) = 0 \Rightarrow q_{1/2} = 0 \]

\[ a_m (m+r)(2(m+r)-1) - a_{m-2} = 0 \]

\[ r = 0: \quad a_m = -a_{m-2} / m (2m-1) \quad m > 2 \]

\[ a_2 = a_0 / 2 (4) = a_0 / 6 \]

\[ a_4 = a_2 / 4 (8) = a_2 / 28 = a_0 / 6.28 \]

\[ y_1 (x) = a_0 x^0 [1 + x^2 / 6 + x^4 / 6.28 + \ldots] \]

\[ r = 1/2 : \quad a_m = a_m / (m+1)(2m+1) \]

\[ a_2 = a_0 / 2.5 \quad a_4 = a_2 / 4.9 = a_0 / 360 \]

\[ y_2 (x) = a_0 x^{1/2} [1 + x^2 / 10 + x^4 / 360 + \ldots] \]
Q3. \[ y = 2x^2 y'' + (3x + x^2) y' - y = 0 \]

\[ \lim_{x \to 0} x (3x + x^2) = 0 = p_0 \quad \lim_{x \to 0} x^2 (-1) = -\frac{1}{2} = q_0 \]

Since \( p_0 < 0 \) and \( q_0 > 0 \), \( x = 0 \) is a regular singular point.

Indicate Eq: \( \tau (r - 1) + \frac{3}{2} r - 2 = 0 \) \( \tau + \frac{1}{2} r = 1 \) \( \tau + \frac{1}{2} (r - 1) = 0 \) \( \tau = -\frac{1}{2} \)

Let \( y = \sum_{n=0}^{\infty} a_n x^n \)

\[ y' = \sum_{n=1}^{\infty} na_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \]

\[ 2x^2 y'' + 3x y' = \sum_{n=2}^{\infty} 2n(n+1) a_n x^{n+2} + \sum_{n=1}^{\infty} 3n a_n x^{n+1} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{2n(n+1)}{n+2} a_n x^{n+2} + \frac{3}{2} n a_n x^{n+1} \right) \]

\[ = \sum_{n=0}^{\infty} \frac{n+1}{n+2} a_n x^{n+2} + \frac{3}{2} a_1 x \]

\[ a_n = \frac{-m+1}{a_1} a_{m-1} \]

\[ a_1 = \frac{-m+1}{a_0} a_m \]

\[ a_n = \frac{-m+1}{a_0} a_{m-1} \]

\[ y_0(x) = a_0 x^{\frac{1}{2}} \left[ 1 - \frac{x}{10} + \frac{3x^2}{280} + \cdots \right] \]

\[ y_1(x) = a_0 x^{\frac{1}{2}} \left[ 1 - \frac{x}{10} + \frac{3x^2}{280} + \cdots \right] \]
4. (a) 0 < x < ∞ are ordinary points, and x = 0 is a singular point.

\[ \lim_{x \to 0} \frac{5x}{6x^3} = \frac{5}{6} = p_0 \quad \lim_{x \to 0} \frac{x^2 (-1)}{6x^3 (1 + x)} = -\frac{1}{6} = q_0 \quad q_0 < 1 < 0 \quad x = 0 \text{ is a regular sing, pt} \]

b) Indicial eq: \( (r+1) + \frac{5}{3} r - \frac{1}{3} = 0 \Rightarrow 6r^2 - r - 1 = (3r+1)(2r-1) = 0 \quad r = -\frac{1}{3}, \frac{1}{2} \)

c) \( y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad y' = \sum_{n=0}^{\infty} \frac{6}{6(n+r)} a_n x^{n+r-1} \quad y'' = \sum_{n=0}^{\infty} \frac{6}{(n+r)(n+r-1)} a_n x^{n+r-2} \)

\[ L_y = 6x^2 y'' = \sum_{n=0}^{\infty} 6(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} \frac{6}{6(n+r)} a_n x^{n+r} \]

\[ \sum_{n=0}^{\infty} \frac{6(n+r)(n+r-1)}{n=m+1} a_m x^{m+r} + \sum_{n=0}^{\infty} \frac{6}{6(n+r)} a_n x^{n+r} = 0 \]

\[ \sum_{m=0}^{\infty} \left\{ (m+r) \left[ 6(m+r+1) + 5 \right] - 1 \right\} a_m x^{m+r} = 0 \]

\[ \omega = \left\{ r^2 (6r^2 - 1) - 1 = (3r+1)(2r-1) = 0 \right\} \quad r = -\frac{1}{3}, \frac{1}{2} \]

\[ x^T \quad 6r^2 - r - 1 = (3r+1)(2r-1) = 0 \quad r = -\frac{1}{3}, \frac{1}{2} \]

\[ x^{m+r} \]

\[ a_m = -\frac{6(m+r-1)(m+r-2)}{(m+r)(6(m+r)-1)} a_{m-1} \]

\[ r = -\frac{1}{3} \quad a_m = -\frac{6(m+4/3)(m-7/3)}{(m-1/3)(6m-2-13/3)} a_{m-1} = -\frac{2}{3} \frac{(3m-4)(3m-7)}{(m-1/3)(3m-1)} a_{m-1} = -\frac{2}{3} (3m-4)(3m-7) a_{m-1} \]

\[ q_1 = \frac{-2(-1)(-4)}{2.1.11} a_0 = -\frac{8}{2.3} a_0 \quad q_2 = \frac{-2(2)(-1)}{3.3.3} a_1 = -\frac{16}{63} a_1 \]

\[ y_1(x) = a_0 x^{1/3} \left[ 1 - \frac{8}{3} x + \frac{16}{63} x^2 + \ldots \right] \]

\[ r = +\frac{1}{2} \quad a_m = -\frac{6(m+1/2)(m-3/2)}{(m+1/2)(6m+2-1)} a_{m-1} = -\frac{3(1/2)(2m-1)(2m-3)}{(2m+1)(3m+1)-1} a_{m-1} \]

\[ q_1 = \frac{-3(1)(-1)}{2.1.11} a_0 = \frac{-3}{22} a_0 \quad q_2 = \frac{-3(3)(1)}{4(17)} a_1 = \frac{-27}{22.4.17} a_1 \]

\[ y_2(x) = a_0 x^{1/2} \left[ 1 + \frac{3}{22} x - \frac{27}{1496} x^2 + \ldots \right] \]
5. a) $0 < x < 0$ are ordinary points & $x = 0$ is a singular point.

\[
\frac{a_n \cdot x^5}{5} = \frac{\frac{5}{6} \cdot x^2 \cdot (1 + x)}{6} = -\frac{1}{6} = q_0 \text{  if 11/6} < 0 \Rightarrow x = 0 \text{ is a regular singular point.}
\]

b) Initial equation:

\[
T(r-1) + \frac{5}{6} T - \frac{1}{6} = 0 \quad 6r^2 - r - 1 = (3r+1)(2r-1) = 0 \quad r = -\frac{1}{3}, \frac{1}{2}
\]

c) \[y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}
\]

\[L\theta = 6x^2 y''' + 5x y' = \sum_{n=0}^{\infty} \frac{6(n+r)(n+r-1)}{n} a_n x^{n+r} + \sum_{n=0}^{\infty} \frac{5(n+r)}{n} a_n x^{n+r} - \sum_{n=0}^{\infty} \frac{5}{n} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0
\]

\[= \sum_{n=0}^{\infty} \left[ (n+r)(6(n+r)+5) - 5 \right] a_n x^{n+r} - \sum_{n=0}^{\infty} \left[ 6(n+r)+5 \right] a_n x^{n+r} = 0
\]

\[= \sum_{n=0}^{\infty} \left[ (n+r) \left( 6(n+r)+5 \right) - 5 \right] a_n x^{n+r} = 0
\]

\[x = \text{As shown}
\]

\[x^{m+r}, m > 1\]

\[\theta_m = \frac{a_{m-1}}{(n+r)\left[ 6(n+r)+5 \right] - 5}
\]

\[r = -\frac{1}{3}: \quad a_m = \frac{a_{m-1}}{(m-\frac{1}{3})(6m-3) - 5} = \frac{a_{m-1}}{6m^2 - 5m + 3} = \frac{a_{m-1}}{m(6m-5)}
\]

\[a_1 = \frac{a_0}{1.1} = a_0 \quad a_2 = \frac{a_0}{2.7} = a_0 \quad a_3 = \frac{a_0}{14}
\]

\[y_1 (x) = a_0 x^{-\frac{1}{3}} \left[ 1 + x + \frac{x^2}{14} + \ldots \right]
\]

\[r = \frac{1}{2}: \quad a_m = \frac{a_{m-1}}{(m+\frac{1}{2})(6m+2) - 5} = \frac{a_{m-1}}{(2m+1)(3m+1) - 5} = \frac{a_{m-1}}{6m^2 + 5m + 2} = \frac{a_{m-1}}{m(6m+5)}
\]

\[a_1 = \frac{a_0}{1.11} = \frac{a_0}{11} \quad a_2 = \frac{a_0}{2.17} = \frac{a_0}{22.17}
\]

\[y_2 (x) = a_0 x^{\frac{1}{2}} \left[ 1 + \frac{x}{11} + \frac{x^2}{22.17} + \ldots \right]
\]