Math 257/316 Assignment 9 Due Friday November 24 in class

Problem 1 (don’t hand in): Find the solution of Laplace’s equation in the semi-infinite strip \( \{(x,y); 0 \leq x \leq 2, y \geq 0\} \) satisfying the following mixed boundary conditions:
\[
\begin{align*}
&u(0,y) = 0, \quad u_x(2,y) = 0 \quad \text{for all } y \geq 0 \\
&u(x,0) = 2 \sin(3\pi x/4) - 3 \sin(7\pi x/4) \quad \text{for all } 0 \leq x \leq 2, \\
&\lim_{y \to +\infty} u(x,y) = 0
\end{align*}
\]

Problem 2 (don’t hand in): A metal plate occupies a quarter-annular region \( 0 < a \leq r \leq b \) and \( 0 \leq \theta \leq \pi/2 \). The vertical face is insulated while the horizontal face and the outer hoop are maintained at 0 degrees. The inner hoop is maintained at a temperature of \( \sin(3\theta) \). Determine the steady state temperature by solving the following BVP in \( \Omega \):
\[
\begin{align*}
&v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta \theta} = 0 \quad \text{in } \Omega \\
v(r,0) = 0 \quad \text{for } a < r < b, \quad v_{\theta}(r,\pi/2) = 0 \quad \text{for } a < r < b \\
v(a,\theta) = \sin 3\theta \quad \text{for } 0 < \theta < \pi/2, \quad v(b,\theta) = 0 \quad \text{for } 0 < \theta < \pi/2
\end{align*}
\]

Problem 3 (don’t hand in): Consider the BVP:
\[
\phi'' + 6\phi' + \lambda \phi = 0, \quad 0 < x < L \\
\phi(0) = 0 \\
\phi(L) = 0
\]
(a) Put this BVP into Sturm-Liouville form.
(b) Compute all eigenvalues and eigenfunctions.
(c) Show explicitly that the eigenfunctions are mutually orthogonal.
(Don’t forget to include the weight function inside the integral.)

Problem 4 (hand in): Consider the eigenvalue problem
\[
x^2 y'' + xy' + \lambda y = 0 \\
y(1) = 0 = y'(2)
\]
(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.
(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace’s equation on the quarter-annular region:
\[
\begin{align*}
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} &= 0, \quad 1 < r < 2, \quad 0 < \theta < \pi/2 \\
u(r,0) &= 0 \quad \text{and} \quad \frac{\partial u(r,\pi/2)}{\partial \theta} = f(r) \\
u(1,\theta) &= 0 \quad \text{and} \quad \frac{\partial u(2,\theta)}{\partial r} = 0
\end{align*}
\]
Problem 5 (don’t hand in):
Solve the following heat conduction problem:

\[
\begin{align*}
    u_t &= x^2 u_{xx} + 4x u_x \quad \text{for } x \in (1, 2), \ t > 0 \\
    u(1, t) &= 1 \quad u(2, t) = 1 \\
    u(x, 0) &= 1 - 5x^{-3/2}
\end{align*}
\]

Problem 6 (hand in): (Seepage from a dam - a soil mechanics problem)

Consider the problem of flow under a wall in a 10 m thick sand layer underlain by an impermeable clay deposit. The water level at the right side of the wall is assumed to be at a height of 2 m above the ground level, and the water level at the left side of the wall is assumed to coincide with the ground surface. Under the influence of this water-level-difference, groundwater will flow under the wall, from right to left. Two sheet piles, located at a distance of 10 m from the wall constrain the flow to a rectangular \(10m \times 20m\) region. See figure 1 below.

The hydraulic head \(h(x, y)\), which is proportional to the fluid pressure, satisfies Laplace’s equation. Along with Laplace’s equation and the appropriate boundary conditions the boundary value problem for \(h(x, y)\) is:
\[
\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad \text{for} \quad -10 < x < 10, \quad -10 < y < 0 \quad (1)
\]

\[
h(x,0) = \begin{cases} 
0 & \text{for} \quad -10 \leq x < 0 \\
2 & \text{for} \quad 0 \leq x \leq 10,
\end{cases}
\]

\[
h_x(-10, y) = 0 \quad \text{and} \quad h_x(10, y) = 0 \quad \text{for} \quad -10 \leq y \leq 0
\]

\[
h_y(x, -10) = 0 \quad \text{for} \quad -10 < x < 10
\]

Download the spreadsheet Laplace Soil_Starter.xls and implement the boundary conditions defined in (1). The basic finite difference stencil is coded into cell E15. You will need to implement the zero flux boundary conditions by adding ghost meshpoints in the magenta colored cells. The Dirichlet boundary condition needs to be implemented along the line segment \( y = 0 \). In order that there is a smooth transition from 2 when \( x > 0 \) to 0 when \( x < 0 \) assume that \( h(0,0) = 1 \).

Now use the iterate feature to determine \( h(x, y) \). Ensure that the solution has reached a steady state and print out the surface plot of \( h(x, y) \) and hand this in with your assignment.