$$u_t = \alpha^2 u_{xx} \quad 0<x<L$$

BC: $u(0,t) = At \quad u(L,t) = 0$

IC: $u(x,t) = 0$

LET $u(x,t) = w(x,t) + v(x,t)$ WHERE $w(x,t) = At\left(1-\frac{x}{L}\right)$

THEN $v_t = \alpha^2 v_{xx} - A\left(1-\frac{x}{L}\right)$

$v(0,t) = 0 = v(L,t)$

$v(x,0) = 0$

LET $S(x,t) = -A\left(1-\frac{x}{L}\right) = \sum_{n=1}^{\infty} \hat{S}_n(t) \sin\left(\frac{n\pi x}{L}\right)$

$$\hat{S}_n = \frac{2}{L}\int_0^L A\left(\frac{x}{L}-1\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{2A}{n\pi}$$

NOW LET $v(x,t) = \sum_{n=1}^{\infty} \hat{v}_n(t) \sin\left(\frac{n\pi x}{L}\right)$

$$v_t = \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t)\sin\left(\frac{n\pi x}{L}\right) \qquad v_{xx} = -\sum_{n=1}^{\infty} \hat{v}_n(t)\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore 0 = v_t - \alpha^2 v_{xx} - S(x,t) = \sum_{n=1}^{\infty}\left\{\dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 \hat{v}_n + \frac{2A}{n\pi}\right\}\sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore \quad \dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 \hat{v}_n(t) = -\frac{2A}{n\pi}$$

$$\left(e^{+\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n(t)\right)^{\bullet} = -\frac{2A}{n\pi} e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

$$e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n(t) = -\frac{2AL^2}{\alpha^2(n\pi)^3} e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t} + B$$

$$\hat{v}_n(t) = -\frac{2AL^2}{\alpha^2(n\pi)^3} + B e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

$$0 = \hat{v}_n(0) = -\frac{2AL^2}{\alpha^2(n\pi)^3} + B$$

$$\therefore \quad \hat{v}_n(t) = \frac{2AL^2}{\alpha^2(n\pi)^3}\left(e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} - 1\right)$$

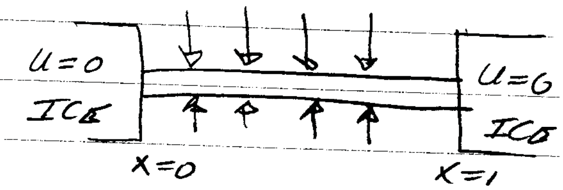
$$\therefore u(x,t) = At\left(1-\frac{x}{L}\right) + \frac{2AL^2}{\pi^3\alpha^2}\sum_{n=1}^{\infty}\frac{\left(e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} - 1\right)}{n^3}\sin\left(\frac{n\pi x}{L}\right)$$

PROBLEM 2: A BAR WITH AN EXTERNAL HEAT SOURCE $S(x)$

$$u_t = \alpha^2 u_{xx} + \sin(3\pi x)$$

BC: $u(0,t) = 0 = u(1,t)$

IC: $u(x,0) = \sin(\pi x)$



METHOD 1: STEADY SOLN $\frac{\partial u_\infty}{\partial t} = 0 \qquad \alpha^2 u_\infty'' + \sin(3\pi x) = 0$

$$\therefore \quad u_\infty' = + \frac{\cos(3\pi x)}{\alpha^2(3\pi)} + A \qquad u_\infty(x) = \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2} + Ax + B$$

$$u_\infty(0) = B = 0 \qquad u_\infty(1) = \frac{\cancel{\sin(3\pi)}^{\,0}}{\alpha^2(3\pi)^2} + A = 0 \Rightarrow u_\infty(x) = \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2}$$

LET $u(x,t) = u_\infty(x) + v(x,t) \Rightarrow (\cancel{u_\infty}^{\,0} + v)_t = \alpha^2(u_\infty + v)_{xx} + \sin(3\pi x)$

$$\therefore \; v_t = \alpha^2 v_{xx}$$

$$0 = u(0,t) = \cancel{u_\infty(0)} + v(0,t) \Rightarrow v(0,t) = 0$$

$$0 = u(1,t) = u_\infty(1) + v(1,t) \Rightarrow v(1,t) = 0$$

$$u(x,0) = u_\infty(x) + v(x,0) = \sin(\pi x) \Rightarrow v(x,0) = \sin(\pi x) - \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2}$$

BY SEPARATION OF VARIABLES

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2(n\pi)^2 t} \sin(n\pi x)$$

WHERE $v(x,0) = \sin(\pi x) - \frac{\sin(3\pi x)}{\alpha^2(3\pi)^2} = b_1 \sin(\pi x) + b_2 \sin(2\pi x) + b_3 \sin(3\pi x) + \ldots$

$$\therefore \; b_1 = 1 \qquad b_3 = -\frac{1}{\alpha^2(3\pi)^2} \qquad b_K = 0 \quad K \neq 1 \text{ OR } 3$$

$$v(x,t) = e^{-\alpha^2\pi^2 t}\sin(\pi x) - \frac{e^{-\alpha^2(3\pi)^2 t}}{\alpha^2(3\pi)^2}\sin(3\pi x)$$

$$u(x,t) = u_\infty(x) + v(x,t)$$

$$= \underset{\uparrow\,\text{STEADY}}{\frac{\sin(3\pi x)}{\alpha^2(3\pi)^2}} + \underbrace{e^{-\alpha^2\pi^2 t}\sin(\pi x) - \frac{e^{-\alpha^2(3\pi)^2 t}}{\alpha^2(3\pi)^2}\sin(3\pi x)}_{\text{TRANSIENT}}$$

$$u(x,t) = e^{-\alpha^2\pi^2 t}\sin(\pi x) + \frac{\left(1 - e^{-\alpha^2(3\pi)^2 t}\right)}{\alpha^2(3\pi)^2}\sin(3\pi x)$$

METHOD 2: USING EIGENFUNCTION EXPANSIONS:

THE APPLICABLE EIGENVALUE PROBLEM IS:

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\} \quad \begin{array}{l} X = A\cos\lambda x + B\sin\lambda x \quad X(0) = A = 0 \\ X(1) = B\sin\lambda = 0 \quad \lambda_n = n\pi \quad n = 1, 2, \ldots \quad X_n = \sin(n\pi x) \end{array}$$

LET $u(x,t) = \sum_{n=1}^{\infty} \hat{u}_n(t) \sin(n\pi x)$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{d\hat{u}_n}{dt} \sin(n\pi x) \qquad \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \hat{u}_n \{-(n\pi)^2 \sin(n\pi x)\}$$

$$\therefore \; u_t - \alpha^2 u_{xx} = \sum_{n=1}^{\infty} \left\{ \frac{d\hat{u}_n}{dt} + \alpha^2 (n\pi)^2 \hat{u}_n \right\} \sin(n\pi x) = \sin(3\pi x)$$

$$\therefore \; \frac{d\hat{u}_n}{dt} + \alpha^2 (n\pi)^2 \hat{u}_n = \left\{ \begin{array}{ll} 1 & n = 3 \\ 0 & n \neq 3 \end{array} \right\} = \delta_{n3}$$

$$\frac{d}{dt}\left(e^{\alpha^2 (n\pi)^2 t} \hat{u}_n \right) = e^{\alpha^2 (n\pi)^2 t} \delta_{n3}$$

$$e^{\alpha^2 (n\pi)^2 t} \hat{u}_n = \frac{e^{\alpha^2 (n\pi)^2 t}}{\alpha^2 (n\pi)^2} \delta_{n3} + C_n$$

$$\hat{u}_n(t) = \frac{\delta_{n3}}{\alpha^2 (n\pi)^2} + C_n e^{-\alpha^2 (n\pi)^2 t}$$

$$\therefore \; u(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{\delta_{n3}}{\alpha^2 (n\pi)^2} + C_n e^{-\alpha^2 (n\pi)^2 t} \right\} \sin(n\pi x)$$

$$= \frac{\sin(3\pi x)}{\alpha^2 (3\pi)^2} + \sum_{n=1}^{\infty} C_n e^{-\alpha^2 (n\pi)^2 t} \sin(n\pi x)$$

NOW $u(x,0) = \sin(\pi x) = \dfrac{\sin(3\pi x)}{\alpha^2 (3\pi)^2} + \sum_{n=1}^{\infty} C_n \sin(n\pi x)$

$$\therefore \; \sin(\pi x) - \frac{\sin(3\pi x)}{\alpha^2 (3\pi)^2} = C_1 \sin(\pi x) + C_2 \sin(2\pi x) + C_3 \sin(3\pi x) + \cdots$$

$C_1 = 1 \quad C_3 = -1/\alpha^2 (3\pi)^2 \qquad C_k = 0 \quad k \neq 1, 3$

$$\therefore \; u(x,t) = e^{-\alpha^2 (n\pi)^2 t} + \frac{\left(1 - e^{-\alpha^2 (3\pi)^2 t}\right)}{\alpha^2 (3\pi)^2} \sin(3\pi x)$$

(4) $u_t = u_{xx}$

$u(0,t) = 0$

$u(\pi,t) = t^2$

$u(x,0) = 0$

a) The function $w(x,t) = t^2 \frac{x}{\pi}$ satisfies the BCs.

Let $v(x,t) = u(x,t) - w(x,t)$

Then $u_t = u_{xx}$

$\Rightarrow (v+w)_t = (v+w)_{xx}$

$\Rightarrow v_t + \frac{2}{\pi} t x = v_{xx}$

$\Rightarrow v_t = v_{xx} - \frac{2}{\pi} t x$

Also $v(0,t) = 0$

$v(\pi,t) = 0$

$v(x,0) = u(x,0) - w(x,0) = 0$

We solve by the method of eigenfunction ~~expansion~~ expansion.

Here eigenfunctions are $\sin(nx)$, $n = 1, 2, 3, \ldots$

Write $v(x,t) = \sum_{n=1}^{\infty} V_n(t) \sin(nx)$

$v_t = v_{xx} - \frac{2}{\pi} t x$

$\Rightarrow \sum_{n=1}^{\infty} V_n'(t) \sin(nx) = \sum_{n=1}^{\infty} V_n(t) (-n^2) \sin(nx) + \sum_{n=1}^{\infty} b_n(t) \sin(nx)$

where $b_n(t) = \frac{2}{\pi} \int_0^{\pi} -\frac{2}{\pi} t x \sin(nx)\, dx$

$= \frac{-4t}{\pi^2} \int_0^{\pi} x \sin(nx)\, dx$

$$= \frac{-4t}{\pi^2}\left(\frac{-1}{n} x\cos(nx)\Big|_0^{\pi} + \cancel{\frac{1}{n}\int_0^{\pi} \cos(nx)\, dx} \right)$$

$$= \frac{-4t}{\pi^2}\,\frac{-1}{n}\left(\pi\cos(n\pi) - 0 \right)$$

$$= \frac{4t}{n\pi}\cos(n\pi)$$

$$= \frac{4t(-1)^n}{n\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[V_n'(t) + n^2 V_n(t) - \frac{4t(-1)^n}{n\pi} \right]\sin(nx) = 0$$

$$\Rightarrow V_n'(t) + n^2 V_n(t) = \frac{4t(-1)^n}{n\pi}$$

$$\Rightarrow \frac{d}{dt}\left(e^{n^2t} V_n(t) \right) = \frac{4t(-1)^n}{n\pi} e^{n^2t} \qquad \left(\begin{array}{c}\text{integrating} \\ \text{factor} \\ \text{method}\end{array}\right)$$

$$\Rightarrow e^{n^2t} V_n(t) = \frac{4(-1)^n}{n\pi}\underbrace{\int t e^{n^2t}\, dt} + C_n$$

$$= \frac{1}{n^2} t e^{n^2t} - \frac{1}{n^2}\int e^{n^2t}\, dt$$

$$= \frac{1}{n^2} t e^{n^2t} - \frac{1}{n^4} e^{n^2t}$$

$$= \frac{1}{n^2} e^{n^2t}\left(t - \frac{1}{n^2} \right)$$

$$\Rightarrow V_n(t) = \frac{4(-1)^n}{n\pi}\,\frac{1}{n^2}\left(t - \frac{1}{n^2} \right) + C_n e^{-n^2t}$$

$$IC: \quad 0 = v(x,0) = \sum_{n=1}^{\infty} V_n(0) \sin(nx)$$

$$\Rightarrow \quad V_n(0) = 0 \quad , \quad \forall n$$

Here $\quad V_n(0) = \frac{4(-1)^n}{n\pi} \frac{1}{n^2}\left(\frac{-1}{n^2}\right) + c_n$

$$\Rightarrow \quad c_n = \frac{4(-1)^n}{\pi n^5}$$

Thus $\quad u(x,t) = w(x,t) + v(x,t)$

$$\boxed{u(x,t) = \frac{1}{\pi} t^2 x + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{\pi n^3}\left(t - \frac{1}{n^2}\right) + \frac{4(-1)^n}{\pi n^5} e^{-n^2 t} \right) \sin(nx)}$$

b) $\quad w(x,t) = \frac{1}{\pi} t^2 x + \frac{1}{3\pi}(x^2 - \pi^2) t x$

$$+ \frac{1}{180\pi}(3x^4 - 10\pi^2 x^2 + 7\pi^4) x$$

$$w(0,t) = 0 + 0 + 0 = 0 \quad ✓$$

$$w(\pi,t) = t^2 + 0 + \frac{1}{180\pi}(3\pi^4 - 10\pi^4 + 7\pi^4)\pi = t^2 \quad ✓$$

$$w_t(x,t) = \frac{2}{\pi} t x + \frac{1}{3\pi}(x^2 - \pi^2) x$$

$$w_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\left(\frac{t}{3\pi} x^3 + \frac{1}{60\pi} x^5 - \frac{\pi}{18} x^3 \right)$$

$$= \frac{2t}{\pi} x + \frac{1}{3\pi} x^3 - \frac{\pi}{3} x$$

$$= \frac{2}{\pi} t x + \frac{1}{3\pi}(x^2 - \pi^2) x$$

$$= w_t(x,t) \quad ✓$$

c) Let $v(x,t) = u(x,t) - w(x,t)$

$$\Rightarrow \quad v_t = v_{xx}$$
$$v(0,t) = 0$$
$$v(\pi,t) = 0$$
$$v(x,0) = u(x,0) - w(x,0)$$
$$= \frac{-1}{180\pi}\left(3x^4 - 10\pi^2x^2 + 7\pi^4\right)x$$

Separation of variables:
$$v(x,t) = X(x)\,T(t)$$

$$v_t = v_{xx}$$
$$\Rightarrow \quad XT' = X''T$$
$$\Rightarrow \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda^2 \quad , \text{ for nontrivial solutions}$$

We have,
$$X'' + \lambda^2 X = 0$$
$$X(0) = 0$$
$$X(\pi) = 0$$

Then $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$

$$X(0) = 0 = A$$
$$X(\pi) = 0 = B\sin(\lambda\pi)$$
$$\Rightarrow \quad \lambda_n = n \quad \text{—— eigenvalues} \quad , n = 1, 2, \ldots$$
$$\Rightarrow \quad X_n(x) = \sin(n\pi) \quad \text{—— eigenfunctions}$$

Also,
$$T' + \lambda^2 T = 0$$
$$\Rightarrow \quad T_n(t) = e^{-n^2 t}$$

Then $$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx)$$

$$v(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

$$\Rightarrow \quad c_n = \frac{2}{\pi}\int_0^{\pi} v(x,0)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{-1}{180\pi}\left(3x^4 - 10\pi^2 x^2 + 7\pi^4\right)x \sin(nx)\,dx$$

$$\cancel{= \frac{-1}{90\pi^2}\left[3\int_0^{\pi} x^5 \sin(nx)\,dx - 10\pi^2\int_0^{\pi} x^3\sin(nx)\,dx + 7\pi^4\int_0^{\pi} x\sin(nx)\,dx\right]}$$

$$= \frac{-1}{90\pi^2}\int_0^{\pi}\left(3x^5 - 10\pi^2 x^3 + 7\pi^4 x\right)\sin(nx)\,dx$$

$$= \frac{-1}{90\pi^2}\left(\frac{-360\pi(-1)^n}{n^5}\right)$$

$$= \frac{4(-1)^n}{\pi n^5}$$

Thus $$u(x,t) = w(x,t) + v(x,t)$$

$$\boxed{u(x,t) = \frac{1}{\pi}t^2 x + \frac{1}{3\pi}(x^2-\pi^2)tx + \frac{1}{180\pi}\left(3x^4 - 10\pi^2x^2 + 7\pi^4\right)x + \sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi n^5}e^{-n^2 t}\sin(nx)}$$

d) The answers appear different but are the same because $\sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi n^3}\left(t - \frac{1}{n^2}\right)$ is the Fourier sine series of

$$\frac{1}{3\pi}(x^2-\pi^2)tx + \frac{1}{180\pi}\left(3x^4 - 10\pi^2x^2 + 7\pi^4\right)x.$$