Problem 1: (Do Not hand in) Fourier Series and Parseval’s Theorem
Let $f^o(x)$ be the odd periodic extension of period $2\pi$ of the function $f(x) = x(\pi - x)$ defined on $[0, \pi]$.

a) Determine the Fourier series for $f^o(x)$.
b) Use this series to show that $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$
c) Now use Parseval’s Theorem to show that $\frac{\pi^6}{960} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$
d) Use the result in (c) to sum the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$. (Hint split this series into even and odd terms).
e) Use the first 4 terms in the series in (b) to determine an approximate for $\pi$. Now use the first 4 terms in the series (c) to determine an approximation to $\pi$. Which approximation is better? Give the results to 6 digits.

Problem 2: (Do not hand in) Steady state solutions: Find the steady-state solutions for the following heat conduction boundary value problems:

a) $u_t = \alpha^2 u_{xx}$, $u(0, t) = 1$, $u(\pi, t) = 2$
b) $u_t = \alpha^2 u_{xx}$, $u(0, t) = 5$, $u_x(1, t) = 0$
c) $u_t = \alpha^2 u_{xx}$, $u(0, t) = 0$, $u_x(2, t) + u(2, t) = 4$
d) $u_t = u_{xx} - \beta^2 u$, $u(0, t) = 1$, $u(\pi, t) = 2$
e) $u_t = u_{xx} - \beta^2 u$, $u_x(0, t) = 1$, $u(\pi, t) = 2$

(see Problems 3 & 4 on page 2)
Problem 3: (Hand in) Designing a fire-resistant beam:
Slim floor beams are fabricated by welding an I- or T-section (half an I-section) to a plate designed to support composite or concrete floors on their bottom flange or plate (see figure above). By use of transversal rebars passing through the beam web, a steel-concrete composite action may be activated and lead to economic designs for spans up to 14 meters.
These beams offer several advantages compared to common solutions: reduced floor thickness, high speed of erection of the structure, and also inherent good fire resistance.
According to European Standards, the Temperature of 400°C is the limit point at which the mechanical properties of the steel start to decrease. Therefore it is useful to have a simple model with an analytic solution that can be used in the design process. In the event of a fire under the slab, only the lower plate is directly subjected to a heat flux resulting from elevated gas temperatures. A finite element analysis (see contour plot) of the temperature distribution after 1 hour shows that the part of the steel section integrated in the slab is heated by conduction only and that the temperature distribution in the steel section is highly non-uniform.

We consider the following model governing the temperature within the conducting I Beam:

\[
\begin{align*}
  u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0 \\
  BC : \quad u_x(0, t) &= 0, \quad u_x(1, t) = 0 \\
  IC : \quad u(x, 0) &= 0
\end{align*}
\]

(a) Determine a solution \( u(x, t) \) to the PDE and boundary conditions that is quadratic in \( x \) and linear in time \( t \).

(b) Let \( u(x, t) = w(x, t) + v(x, t) \) and identify the PDE, BC and IC satisfied by \( v(x, t) \).

(c) Use the method of separation of variables to solve the above boundary value problem for \( v(x, t) \) and from this determine the solution \( u(x, t) \).

(d) **Excel Exercise:** The flux boundary condition at the left and right endpoints of the bar can be approximated by the following difference quotients:

\[
\begin{align*}
  \frac{\partial u(1, t)}{\partial x} &= \frac{u(x_N + \Delta x, t) - u(x_N - \Delta x, t)}{2\Delta x} = 0 \\
  \frac{\partial u(0, t)}{\partial x} &= \frac{u(\Delta x, t) - u(-\Delta x, t)}{2\Delta x} = 1
\end{align*}
\]

Implement these boundary conditions in the spreadsheet: **Heat0.xls** posted on the web site by inserting fictitious mesh points in two extra columns. Plot \( u(x, t = 0.5) \) obtained using the numerical solution, print it out and hand it in with your assignment.

**Problem 4: (Hand in) Solving a PDE with a time independent source using the steady state solution:**

Consider the following boundary value problem for the heat equation governing the temperature within a conducting bar:

\[
\begin{align*}
  u_t &= u_{xx} + 6(\pi - x), \quad 0 < x < \pi, \quad t > 0 \\
  BC : \quad u(0, t) &= 0 = u(\pi, t) \\
  IC : \quad u(x, 0) &= x^3 - 3\pi x^2
\end{align*}
\]

(a) Determine the steady-state temperature \( u_\infty(x) \).

(b) Let \( u(x, t) = u_\infty(x) + v(x, t) \) and identify the PDE, BC and IC satisfied by \( v(x, t) \).

(c) Use the method of separation of variables to solve the above boundary value problem for \( v(x, t) \) and from this determine the solution \( u(x, t) \).