1. \( u_{tt} = \alpha^2 u_{xx} \) (1)

**BC:** \( \frac{\partial u(0,t)}{\partial x} = 0 \quad \frac{\partial u(\pi,t)}{\partial x} = 0 \)

**IC:** \( u(x,0) = \cos \gamma x \quad 0 < x < \pi \)

Let \( u(x,t) = \bar{x}(x) \bar{T}(t) \Rightarrow \bar{T}/\alpha^2 \bar{T}(t) = \bar{x}''(x)/\bar{x}(x) = \text{const} = -\lambda^2 \)

\( \Rightarrow \bar{T} = -\alpha^2 \lambda^2 \bar{T} \Rightarrow \bar{T}(\pi) = C e^{-\pi^2 \lambda^2} \)

\( \bar{x}'' + \lambda^2 \bar{x} = 0 \) \( \Rightarrow \bar{x}(x) = A \cos \lambda x + B \sin \lambda x \)

\( \bar{x}'(0) = 0 = \bar{x}'(\pi) \)

\( \bar{x}'(0) = \Rightarrow B = 0 \Rightarrow \lambda = n, n = 1, 2, \ldots \)

\( \bar{x}'(\pi) = -A \lambda \sin \lambda \pi = 0 \Rightarrow \lambda = n, n = 1, 2, \ldots \)

Thus, \( \lambda_n = n, n = 1, 2, \ldots \) are the eigenvalues and \( \bar{x}_n = \cos (nx) \) are the corresponding eigenfunctions.

\( \lambda = 0 \)

\( \bar{x} = Ax + B \quad \bar{x}' = A \)

\( \bar{x}'(0) = A = 0 \) so that the BC \( \bar{x}'(\pi) \) holds automatically.

\( \therefore \bar{x}_0 (x) = B \cdot 1 \) is the non-trivial eigenfunction.

Corresponding to \( \lambda = 0 \)

By superposition (because (i) is linear)

\( u(x,t) = \sum_{n=1}^{\infty} \bar{x}_n \cos (nx) e^{-\pi^2 n^2 t} \)

Now \( \cos \gamma x = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos (nx) \)

which is just a half-range Fourier cosine series \( A_0 = \frac{1}{\pi} \int_{0}^{\pi} \cos \gamma x \, dx \)

\( \Rightarrow A_0 = \frac{2}{\pi} \int_{0}^{\pi} \sin \gamma x \, dx \)

\( \Rightarrow A_0 = \frac{2}{\pi} \left[ \frac{\sin (\gamma \pi)}{\gamma \pi} \right] \)

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\( \Rightarrow A_n = \frac{2}{\pi} \int_{0}^{\pi} \cos \gamma x \cos (nx) \, dx = \begin{cases} 0 & \gamma \neq n \\ 1 & \gamma = n \end{cases} \)

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\( \Rightarrow \gamma = 0 \in \mathbb{Z} : \quad u(x,t) = 1 \)

\( \gamma = n \in \mathbb{Z} : \quad u(x,t) = e^{-\pi^2 n^2 t} \cos (nx) \)

\( \gamma \notin \mathbb{Z} : \quad u(x,t) = \sum_{n=1}^{\infty} \frac{\sin (\gamma \pi n)}{(\gamma \pi n)} + \sum_{n=1}^{\infty} \frac{\sin ((\gamma - n) \pi n)}{(\gamma - n) \pi n} \cos (nx) e^{-\pi^2 n^2 t} \)
\[ u(t, 0) = \cos \frac{\pi t}{\tau} = \frac{\sin \frac{\pi t}{\tau} + 2 \sum_{n=1}^{\infty} (\frac{\pi t}{\tau} + n^2)}{\pi} \left( \sin \frac{\pi t}{\tau} \cos \frac{\pi t}{\tau} + \cos \frac{\pi t}{\tau} \sin \frac{\pi t}{\tau} \right) \cos \frac{\pi t}{\tau} \]

\[
= \frac{\sin \frac{\pi t}{\tau} + 2 \sum_{n=1}^{\infty} (\frac{\pi t}{\tau} + n^2)}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{\pi t}{\tau} \cos \frac{\pi t}{\tau} + \cos \frac{\pi t}{\tau} \sin \frac{\pi t}{\tau}}{\frac{\pi t}{\tau} + n^2} \]

\[
= \frac{\sin \frac{\pi t}{\tau} + 2 \sum_{n=1}^{\infty} (\frac{\pi t}{\tau} + n^2)}{\pi} \sum_{n=1}^{\infty} \frac{2 \frac{\pi t}{\tau}}{\frac{\pi t}{\tau} + n^2 - \frac{\pi t}{\tau}}
\]

\[
\therefore \quad \cot \frac{\pi t}{\tau} = \frac{1}{\pi} \left\{ \frac{1}{\frac{\pi t}{\tau}} - \sum_{n=1}^{\infty} \frac{2 \frac{\pi t}{\tau}}{\frac{\pi t}{\tau} - n^2} \right\}
\]

Graph for 2 (i) using 5 terms:

![Graph of a function and its Fourier Series](image1)

Graph for 2 (ii) using 5 terms:

![Graph of a function and its Fourier Series](image2)
\( a_0 = \frac{1}{\pi} \int_0^\pi x^2 \, dx = \frac{1}{\pi} \left( \frac{\pi^3}{3} \right) = \frac{\pi^2}{3} \)

\( a_n = \frac{1}{\pi} \int_0^\pi x^2 \cos(nx) \, dx = \frac{1}{\pi} \left[ \frac{x^2 \sin(nx)}{n} \bigg|_0^\pi - \frac{2}{n^2} \int_0^\pi \cos(nx) \, dx \right] \)

\[ = \frac{1}{\pi} \left[ \frac{2x \cos(nx)}{n} \bigg|_0^\pi - \frac{2}{n^2} \int_0^\pi \cos(nx) \, dx \right] \]

\[ = \frac{1}{\pi} \left[ \frac{2\pi (-1)^n - 0 \cdot 1}{n^2} - \frac{2}{n^2} \int_0^\pi \cos(nx) \, dx \right] = 2(-1)^n / n^2 \]

\( b_n = \frac{1}{\pi} \int_0^\pi x^2 \sin(nx) \, dx = \frac{1}{\pi} \left[ \frac{-x^2 \cos(nx)}{n} \bigg|_0^\pi + \frac{2}{n} \int_0^\pi \cos(nx) \, dx \right] \)

\[ = \frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3} \left[ \frac{\sin(nx)}{n} \bigg|_0^\pi - \frac{1}{n^2} \int_0^\pi \sin(nx) \, dx \right] \]

\[ = \frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3} \left[ (-1)^n - 1 \right] \]

\[ \therefore \quad f(x) \sim \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n^3} \left[ (\sin nx) + \cos nx \right] \]

Now \( x = 0 \Rightarrow 0 = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + 0 \Rightarrow \pi^2 = 1 - \frac{1}{2} + \frac{1}{3} - ... \)

\( \therefore \quad x = \frac{\pi}{2} \Rightarrow S \left( \frac{\pi}{2} \right) = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} + 0 \Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2} + \frac{1}{3} + ... \)

(ii) \( \frac{e^x}{x} \)

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^\pi e^x \, dx = \frac{1}{\pi} \left( e^\pi - e^{-\pi} \right) = 2 \sinh \pi / \pi \]

Now \( a_n = \frac{1}{\pi} \int_{-\pi}^\pi e^x \cos nx \, dx = \text{Re} \left[ \int_{-\pi}^\pi e^{(1+in)x} \, dx \right] \)

\( \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^\pi e^x \sin nx \, dx = \text{Im} \left[ \int_{-\pi}^\pi e^{(1+in)x} \, dx \right] \)

\[ \text{Non} \quad \int_{-\pi}^\pi e^{(1+in)x} \, dx = \frac{e^{(1+in)x}}{1+in} \bigg|_{-\pi}^{\pi} = \left( e^{\pi(1+in)} - e^{-\pi(1+in)} \right) (1+in) \]

\[ = [e^{\pi}(-i)^n - e^{-\pi}(-i)^n] (1-in)/(1+n^2) = 2 \sinh \pi (-1)^n (1-in)/(1+n^2) \]

\[ \therefore \quad f(x) = \frac{\sinh \pi}{\pi} \left[ \frac{1}{1+\frac{1}{n^2}} \sum_{n=1}^{\infty} (-1)^n \frac{(\cos nx - n \sin nx)}{1+n^2} \right] \]
3. (a) \( f(-x) = (-x)^2 + 1-x^1 = x^2 + 1\cdot x^1 = f(x) \) \( \text{ f is even.} \)
(b) \( f(-x) = e^{\sin^2(-x)} = e^{\sin^2 x} = f(x) \) \( \text{even.} \)
(c) \( f(-x) = \cosh(-x) + \sinh(-x) = \cosh x - \sinh x \neq f(x) \) \( \text{Neither} \)

4. (a) \( f(x) = x^2 \) \( 0 \leq x \leq \pi \)

(i) **Half-Range Cosine Series** - we want the even extension of \( f \):

\[
b_n = 0
\]

\[
a_0 = \frac{2}{\pi} \int_0^\pi x^2 \, dx = \frac{2}{\pi} \frac{x^3}{3} \bigg|_0^\pi = 2\pi^2 / 3
\]

\[
f_{\text{even}}(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}
\]

\[
a_n = 2 \left\{ \frac{2(-1)^n}{\pi^2} \right\} = 4(-1)^n / \pi^2
\]

Using integral in 2(ii)

(ii) **Half-Range Sine Series** - we need the odd extension of \( f \):

\[
a_n = 0
\]

\[
b_n = 2 \int_0^\pi x^2 \sin(nx) \, dx = 2 \left\{ \frac{\pi(-1)^{n+1}}{n \pi^3} + \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n n^{-1} - 1}{n} \sin(nx) \right\}
\]

\[
f_{\text{odd}}(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n \pi^3} \sin(nx)
\]

(b)

\[
a_0 = \frac{1}{\pi / 2} \int_{-\pi/2}^{\pi/2} f(x) \, dx = \frac{2}{\pi} \int_0^\pi x^2 \, dx = \frac{2}{\pi} \frac{\pi^3}{3} = 2\pi^2 / 3
\]

\[
a_n = \frac{2}{\pi} \int_0^{\pi/2} x^2 \cos(n\pi x) \, dx = \frac{2}{\pi} \int_0^{\pi/2} x^2 \cos(2nx) \, dx
\]

\[
= \frac{2}{\pi} \left[ \frac{2x \cos(2nx)}{(2n)^2} \right]_{0}^{\pi/2} = \frac{2}{(2n)^3} \sin(2nx) \bigg|_0^{\pi/2} = \frac{1}{n^2} \quad (\text{using 2(i)})
\]

\[
b_n = \frac{2}{\pi} \int_0^{\pi/2} x^2 \sin(2nx) \, dx
\]
\[b_n = 2 \left\{ -\frac{\pi}{2(n)} + \frac{2}{\pi^3} \cos(2nx) \right\}^\pi_0 = -\frac{\pi}{n} \quad \text{using previous integrals from 2(i)} \]

\[\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} - \pi \frac{\sin(nx)}{n}\]