Math 257/316 PDE  Assignment 4  
Due in class on Wednesday October 7, 2015

1. **Separation of variables**: Determine whether the method of separation of variables can be used to replace the following PDE’s by a pair of ODE’s. If so, find the equations.
   (a) \(x^2u_{xx} = tu_t\).  
   (b) \(u_{xx} + (x + y)u_{yy} = 0\).

2. **Eigenvalue Problems**: Find all eigenvalues and corresponding eigenfunctions for the following problems
   (a) \(y'' + \lambda y = 0, \quad 0 < x < 1\), \(y(0) = 0, \quad y(1) = 0\).
   (b) \(y'' + 2y' + \lambda y = 0, \quad 0 < x < \pi\), \(y(0) = 0, \quad y(\pi) = 0\).

3. **Finite Difference Approximations**: Use Taylor’s series about the point \(x\) for \(f(x - \Delta x), f(x + \Delta x),\) and \(f(x + 2\Delta x)\) to determine the order \(p\) in each of the following finite difference approximations
   (a) \(-\frac{3f(x) + 4f(x + \Delta x) - f(x + 2\Delta x)}{2\Delta x} = f'(x) + O(\Delta x^p)\)
   (b) \(\frac{f(x) - 2f(x + \Delta x) + f(x + 2\Delta x)}{\Delta x^2} = f''(x) + O(\Delta x^p)\)

4. **Flux boundary condition**: Consider the following boundary value problem for the heat equation
   \[ u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad \alpha^2 = 0.2 \]
   \[ BC : u(0, t) = 0, \quad u_x(1, t) = 0 \]
   \[ IC : u(x, 0) = \sin \left(\frac{3\pi x}{2}\right) \]
   (a) Use the method of separation of variables to solve the above boundary value problem.
   (b) **EXCEL EXERCISE**: As shown in class the insulated boundary condition at the right endpoint of the bar \(x = 1\) can be approximated by the following difference quotient:
   \[ \frac{\partial u(1, t)}{\partial x} = \frac{u(x_N + \Delta x, t) - u(x_N - \Delta x, t)}{2\Delta x} = 0 \]
   This equation reduces to the condition: \(u(x_N + \Delta x, t) = u(x_N - \Delta x, t)\). Now if \(x_N = 1\) is the right endpoint of the bar, then \(x_N + \Delta x = 1 + \Delta x\) which falls outside the bar! However, we can trick the finite difference scheme into imposing this boundary condition by introducing a fictitious meshpoint \(x_{N+1} = x_N + \Delta x\) and forcing the value of the solution \(u(x_N + \Delta x, t)\) at this point to be the same as \(u(x_N - \Delta x, t)\) in accordance with the condition above. Implement this in the spreadsheet: **Heat0.xls** posted on the web site by placing these fictitious values in column W. Plot \(u(x, t = 0.5)\) obtained using the numerical solution and that obtained by separating variables on the same plot, print it out and hand it in with your assignment.