Monitoring Hydraulic Fractures: State Estimation using an Extended Kalman Filter

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Abstract

There is considerable interest in using remote elastostatic deformations to identify the evolving geometry of underground fractures that are forced to propagate by the injection of a high pressure viscous fluid. These so-called hydraulic fractures are used to increase the permeability in oil and gas reservoirs as well as to pre-fracture ore-bodies in preparation for a mining process known as block-caving. The undesirable intrusion of these hydraulic fractures into environmentally sensitive areas or into regions in mines which might pose safety hazards, has stimulated the search for techniques to enable the evolving hydraulic fracture geometries to be monitored. Previous approaches to this problem have involved the inversion of the elastostatic data at isolated time steps in the time series provided by tiltmeter measurements of the displacement gradient field at selected points in the elastic medium. At each time step, parameters in simple static models of the fracture (e.g., a single displacement discontinuity or low order moments of the fracture opening) are identified. The approach adopted in this paper is not to regard the sequence of sampled elastostatic data as independent, but rather to treat the data as linked by
the coupled elastic-lubrication equations that govern the propagation of the evolving hydraulic fracture. We combine the Extended Kalman Filter (EKF) with features of a recently developed implicit numerical scheme to solve the coupled free boundary problem in order to form a novel algorithm to identify the evolving fracture geometry. Numerical experiments demonstrate that, despite excluding significant physical processes in the forward numerical model that substantially alter the geometry of the evolving hydraulic fracture, the EKF-numerical algorithm is able to compensate for the un-modeled processes by using the information fed back from tiltmeter data. Indeed, the proposed algorithm is able to provide reasonably faithful estimates of the fracture geometry, which are shown to converge to the actual hydraulic fracture geometry as the number of tiltmeters is increased. The location of tiltmeters can affect the resolution of the technique, which opens the possibility of using the algorithm to design the deployment of tiltmeters to optimize the resolution in regions of particular interest.

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1 Introduction

Hydraulic fractures (HF) are discontinuities induced to propagate in brittle materials by the injection of a viscous fluid. They occur naturally when pressurized magma from deep underground caverns form vertical HF driven by buoyancy forces, which can in turn form geological structures such as dykes and sills [2, 3]. HF are also used in engineering applications, for example to increase the fracture networks in ore-bodies to enhance the the block-caving process used in the mining industry [4, 5]. Perhaps the most ubiquitous engineering application of HF is the deliberate creation of fractures in reservoirs for the enhanced recovery of oil or gas [1, 6].

The propagation of HF into undesirable locations can have severe safety consequences in the mining industry and can cause considerable loss of hydrocarbons and environmental damage in the oil industry. Thus in order to improve the efficacy of HF in industrial applications it is desirable to
improve fracture placement by the adjustment of the available engineering parameters, such as the pump schedule or the properties of the viscous fluid. To this end, there has been considerable research effort devoted to understanding the multi-scale behavior of propagating HF. This has led to a number of mathematical models of varying complexity along with asymptotic solutions for simplified geometries and various propagation regimes [7, 3, 9, 10, 11, 12, 13]. In addition, sophisticated numerical algorithms have been developed [14, 29, 16, 15], to model the evolution of HF with irregular geometries due to the layering of the elastic strata or the presence of regions with discontinuous geological stresses.

With regard to the feedback and monitoring of the evolution of HF there is a paucity of information. The quantities that are readily available in a typical fracture treatment include the volume of fluid pumped and the wellbore pressure. Tiltmeters located in the well-bore itself, in neighboring offset boreholes, or on the earth’s surface are often used to monitor the strain gradient field that is induced by the propagating HF [30]. More recently, a combination of tiltmeter measurements with microseismic images produced by downhole and surface mounted geophones has been employed [33].

Inversion of the tiltmeter time series has thus-far involved two strategies. First, the fracture evolution is regarded as a sequence of equilibrium states in which the tilt measurements at each time-step are used to solve the shape identification problem by inversion of the crack elasticity operator. Since the elliptic elasticity operator smooths the information about the fracture, it can be demonstrated [32] that the typically small number of remote tilt measurements can at best provide reasonably accurate estimates of the first two moments of the fracture opening, while the higher order moments are subject to significant errors. The fracture volume can be determined from the zeroth moment while the first moments can be used to detect asymmetries in the fracture geometry. Shape identification from these noisy higher order moments is an ill-posed inverse problem. In the second approach, the time series has been used to significantly enrich the data in the elastic inversion process [31] using a Bayesian technique to select the best model from a variety of simple models of the fracture. Hitherto, forward models, which include fluid-elastic coupling, have not been combined with the tilt measurement time series in order to achieve more accurate estimates of the fracture geometry. This paper is aimed at incorporating the coupled forward model in the inversion process.

The inversion of elasto-static data with limited tilt measurements is a
classically ill-posed inverse problem in which only limited information about the evolving HF can be determined. However, this approach, which treats each step of the time series as independent of the others, ignores an important component of the problem. Indeed, the fracture configurations at two consecutive time-steps are related by the coupled system of integro-partial differential equations that describe the propagation of hydraulic fractures and determined by the current parameters such as the volume and rheology of the fluid pumped over the time-step. The fact that the coupled dynamic model provides a means of relating the very limited individual tilt measurement data substantially improves the prospects for accurate inversion. For this class of problem the inversion algorithm is now able to exploit all the causally admissible tilt measurements. This substantially mitigates the severe lack of data suffered by the individual elasto-static inverse problems.

In an ideal situation, the coupled forward HF model would capture all the relevant physical processes governing the fracture evolution and all the parameters required by the model would be known precisely. In this case, the forward model could be used to provide the required prediction of the fracture evolution. However, the parametric uncertainty in practical field situations combined with the approximations that have to be made in order to obtain a tractable model, render the direct use of the forward model for fracture prediction impracticable. The result of these modeling errors will manifest as discrepancies between the measured tilts and those that are consistent with the forward model.

This paper explores the possibility of feeding these tilt discrepancies back to the forward model in order that it can compensate for the parametric uncertainties and un-modeled dynamics. We use an Extended Kalman Filter (EKF) in conjunction with a forward numerical model to provide an algorithm to identify the boundary points of the fracture and the fracture width. Since the EKF is designed for the extraction parameters or state variables from time series, this aspect of the paper is not new. The novelty of the results presented here derives from the application of the EKF to this particular class of highly nonlinear free boundary problems, which occur in the modeling of HF. Typically such evolution models are susceptible to uncertainties in the initial conditions, the model parameters (e.g. the elastic moduli or the in situ stress field), and to so-called un-modeled dynamics (e.g. physical processes that have been excluded from the modeling process such as fluid leak-off). We demonstrate that the EKF is able to assimilate the low order moment information intrinsic to the tilt measurement time series and feed
this information back to the numerical model so that, in spite of parametric uncertainties or un-modeled dynamics, the model is able to estimate the fracture geometry and opening with reasonable precision. We also demonstrate how the tilt placement affects the resolution of the identification algorithm.

In section 2 we describe the forward hydraulic fracture model comprising a system of integro-partial differential equations along with a free-boundary problem, discuss its non-dimensionalization, and outline a numerical solution algorithm to locate the free boundary; in section 3 we present the details of the proposed Extended Kalman Filter Numerical (EKFN) algorithm; in section 4 we provide the results of three of numerical experiments chosen to demonstrate the efficacy of the proposed method; in section 5 we provide some concluding remarks.

2 Hydraulic Fracture Modeling

2.1 Governing Equations and Boundary Conditions

While there are a number of analytic and numerical models of HF with varying complexity [1], in order to test the proposed filtering algorithm we choose a relatively simple model comprising a hydraulic fracture propagating in a state of plane strain - also known as the KGD model. This model is chosen as it is possible to incorporate a number of important propagation modes and physical effects, such as spatially varying and even discontinuous in situ stress fields and Carter leak-off while maintaining a relatively modest computational cost.

The elasticity equations in the model which follows, expresses the fact that the pressure exerted by the fluid on the crack surfaces should be in static equilibrium with deformation of the elastic medium and the ambient geological stresses. It is assumed that the time scale on which the fracture is propagating is much slower than that of the elastodynamic transients, which rapidly decay to equilibrium. The model could be formulated in terms of Navier’s equilibrium partial differential equations for the two-dimensional plane. The direct numerical approximation of these equations for a propagating crack by the finite element method, for example, would require expensive re-meshing as the fracture evolves in addition to the computational burden that volume discretization would involve. To avoid this considerable computational expense, we have chosen an equivalent formulation of the crack
propagation problem in terms of an integral equation that only requires discretization along the crack surface. While this formulation is limited to piecewise homogeneous, orthotropic, linear elastic materials, we believe that it is sufficient to illustrate the efficacy of the algorithm we are proposing.

Figure 1: The geometry of the fluid-driven one dimensional fracture which is assumed to be in a state of plane strain within an elastic medium

The Elasticity Equation
Since we assume a state of plane strain, the model comprises the following
hypersingular integral equation (see for example [18]):

\[ p(x,t) = -\frac{E'}{4\pi} \int_{\ell_l}^{\ell_r} \frac{w(x',t)}{(x-x')^2} dx' \]  

(1)

This equation expresses the pressure response \( p(x,t) \) of the two-dimensional infinite elastic medium due to a one-dimensional crack with opening represented by \( w(x,t) \) and which occupies the interval \( x \in (\ell_l, \ell_r) \) (see figure 1). The origin of the coordinate system is assumed to be located at the tip of the well bore, while the fracture is assumed to propagate along the \( x \)-axis.

Here \( E' = \frac{E}{1-\nu^2} \), and \( E \) and \( \nu \) are the Young’s modulus and Poisson’s ratio of the surrounding rock, respectively. Moreover, the pressure response is balanced by the so-called net pressure, i.e. \( p = p_f - \sigma_c \) defined to be the difference between the fluid pressure \( p_f \) and the minimum \textit{in situ} confining stress \( \sigma_c \), which is assumed to be orthogonal to the fracture line. The fracture footprint, which is the primary target of the inversion process, is delimited in this context by the evolving extremities \( \ell_l(t) \) and \( \ell_r(t) \) of the fracture. In addition, the complete geometry of the fracture in this context is determined by \( w(x,t) \).

**Fluid Flow Equation**

Fluid flow within the crack is governed by the continuity equation and Poiseuille’s law

\[
\frac{\partial w}{\partial t} + \frac{\partial q}{\partial x} = \frac{C' H(t-t_0(x))}{\sqrt{t-t_0(x)}} + Q(t)\delta(x), \quad q = -\frac{w^3 \partial p_f}{\mu'} \frac{\partial}{\partial x}, \quad \ell_l(t) < x < \ell_r(t)
\]

(2)

where \( q \) represents the volumetric flux of fluid per unit length in the out-of-plane direction.

This system of equations can also be derived directly from the Navier-Stokes equations, by neglecting the inertial terms and assuming that the vertical dimension of the fracture cavity, i.e., \( w \) is much smaller than the length of the fracture \( |\ell_l(t)| \) or \( \ell_r(t) \). We also assume that the fluid and fracture fronts coincide, which is typically true for large confinement situations. Here \( \mu' = 12\mu \), and \( \mu \) is the dynamic fluid viscosity; \( C' = 2C_L \) where \( C_L \) is Carter’s leak-off coefficient; \( t_0(x) \) is the time that the fracture front passes the point \( x \) and \( H(t) \) is the Heaviside function; and \( \delta \) is the Dirac delta function representing a point source at the well bore and \( Q \) is the fluid injection rate.
Boundary and propagation conditions

In addition to the elasticity and lubrication equations (1) and (2), the boundary conditions required to determine the unknown width and pressure fields are that the width should vanish at the fracture tips

\[ w(\ell^l, t) = w(\ell^r, t) = 0 \]  

and that there is no fluid loss from the fracture tips, so that the fluid flux \( q \) should vanish at each tip, i.e.

\[ \lim_{x \to \ell^{l,r}} w^3 \frac{\partial p_f}{\partial x} = 0 \]  

If the locations of the fracture tips (\( \ell^l \) and \( \ell^r \)) are known, then equations (1)-(4) are sufficient to determine the pressure and width fields. However, the tip locations are not known \textit{a priori} so an additional condition needs to be prescribed at each tip in order to locate the boundary points of the fracture. This condition typically requires that the stress intensity factor \( K_I \) at each tip be in limit equilibrium with the local fracture toughness \( K_{IC} \) of the rock in accordance with linear elastic fracture mechanics [8]. This condition can be expressed in terms of the following convenient asymptotic relation:

\[ w \sim K' \frac{\sqrt{|\ell - x|}}{E' \sqrt{|\ell^l - x|}} \]  

where \( K' = 4\left(\frac{2}{\pi}\right)^{1/2} K_{IC} \) is the modified stress intensity factor. The tip asymptotic behavior can be used to determine the location of the fracture front. For example, if, in a numerical scheme, a trial fracture width is known at a collocation point or node close to the tip, then the tip position can be determined by inverting the relation (5).

Equations (1-5) are sufficient to completely determine the unknown field variables \( w \) and \( p_f \) as well as the locations of the fracture free boundary points \( \ell^l \) and \( \ell^r \).

Modes of propagation

The tip asymptote (5) applies if the elastic material ahead of the crack is bonded, which gives rise to the material parameter known as the toughness \( K_{IC} \). If, however, the crack were propagating along the interface between two de-bonded elastic half-spaces, the crack tip asymptote would be different from the square root behavior given in (5). The appropriate asymptotic behavior
can be established by considering a moving coordinate system \( \hat{x} = \ell(t) - x \) and defining new field variables \( w(x, t) = \hat{w}(\ell(t) - x) \) and \( p(x, t) = \hat{p}(\ell(t) - x) \). There is no loss of generality is assuming that \( \ell(t) = \ell(t) \). Ignoring the source and leak-off terms, the fluid-flow equations reduce to the following form

\[
\ell \frac{\partial \hat{w}}{\partial \hat{x}} = \frac{\partial}{\partial \hat{x}} \left( \hat{w}^3 \frac{\partial \hat{p}}{\partial \hat{x}} \right)
\]

(6)

Integrating once and introducing the front velocity \( V = \dot{\ell}(t) \) the lubrication equation reduces to the following form

\[
\mu' V = \hat{w}^2 \frac{\partial \hat{p}}{\partial \hat{x}}
\]

Now assuming that the width behaves as a power law \( \hat{w} \sim A \hat{x}^\alpha \) as \( \hat{x} \to 0 \) it can be shown [7] that

\[
\hat{p} \sim \frac{E'}{4} A \alpha \cot(\pi \alpha) \hat{x}^{\alpha - 1}
\]

Substituting these two asymptotic relations into (6) we obtain the following equation

\[
\frac{4}{\alpha(\alpha - 1) \cot(\pi \alpha)} \left( \frac{\mu' V}{E'} \right) = A \hat{x}^{3\alpha - 2}
\]

In order that the right side of this equation should match the constant left side, we require that \( \alpha = 2/3 \) and, solving for \( A \), we obtain the leading asymptotic behavior for the width

\[
w \sim \beta \left( \frac{\mu' V}{E'} \right)^{1/3} \hat{x}^{2/3}
\]

where \( \beta = 2^{1/3} 3^{5/6} \)

(7)

which was first established by [9]. We observe that the primary mechanism for the dissipation of energy for the de-bonded case is that due to driving the viscous fluid through the crack itself.

If the elastic half-spaces are weakly bonded, a situation which corresponds to a small toughness, then the propagation condition (5) will still be active right at the tip, however, depending on the relative magnitudes of the physical parameters \( K' \) and \( \mu' \) the asymptotic behavior can change from (5) to (7) as one moves away from the tip. Thus in spite of the prescribed propagation condition (5), the asymptotic behavior that is relevant for actually
locating the fracture front (and hence the propagation mode) is determined
by the dominant physical process active at the computational length scale.
Indeed, depending on the relative magnitudes of the physical parameters $K'$,
$\mu'$, and $C'$, respectively the toughness, viscosity, or leak-off may dominate
the behavior of the solution at different distances from the tip. In this case
the toughness asymptote (5) is still present, but the extent of the region over
which it applies is much smaller than the computational length scale, so that
its effect not only cannot be captured without significantly refining the mesh,
but it can also be discarded as subdominant.

The dominant physical process relevant for propagation also corresponds
to the dominant mechanism for the dissipation of energy: if the energy re-
quired to break the rock is dominant then propagation will be in the tough-
ness dominated regime, whereas if the energy dissipation due viscous flow is
dominant then propagation will be in the viscosity dominated regime. Fur-
ther details of this multiscale behavior and its impact on numerical schemes
for hydraulic fracture propagation can be found in [29] and references therein.

2.2 Non-dimensional form of the model equations

In order to reduce the forward model (1)-(5) to non-dimensional form we
introduce the following scaled variables

$$
x = \xi \chi, \quad t = t_* \tau, \quad \ell(t) = \ell_* \gamma(\tau), \quad p = p_* \Pi, \quad w = w_* \Omega, \quad \sigma_c = p_* \Sigma_0 \phi(\chi).
$$

Here $\ell_*$, $t_*$, $p_*$ and $w_*$ represent the characteristic length, time, pressure and
widths that are active in the problem, respectively. The quantities $\chi$, $\tau$, $\gamma$,
$\Pi$, and $\Omega$ represent the dimensionless spatial coordinate, the dimensionless
time, fracture front locations in dimensionless coordinates, the dimensionless
pressure, and the dimensionless width, respectively. These transformations
reduce the elasticity equation to the following dimensionless form

$$
\Pi_f(\chi, \tau) - \Sigma_0 \phi(\chi) = -\frac{G_e}{4\pi} \int_{\gamma'}^{\gamma} \frac{\Omega(\chi', \tau)}{(\chi - \chi')^2} d\chi',
$$

By eliminating the flux, the continuity equation and Poiseuille’s law can be
combined to yield the following lubrication equation, which can be expressed
in the following dimensionless form
\[
\frac{\partial \Omega}{\partial \tau} = \frac{1}{G_m} \frac{\partial}{\partial \chi} \left( \Omega^3 \frac{\partial f}{\partial \chi} \right) + \frac{G_c H(\tau - \tau_0(\chi))}{\sqrt{\tau - \tau_0(\chi)}} + \psi(\tau) G_e \delta(\chi), \quad \gamma^l(\tau) < \chi < \gamma^r(\tau)
\]
and the boundary and propagation conditions are reduced to the following form
\[
\Omega(\gamma^{lr}, \tau) = 0, \quad \lim_{\chi \to \gamma^-} \Omega^3 \frac{\partial f}{\partial \chi} = 0, \quad \lim_{\chi \to \gamma^+} \Omega = G_k \sqrt{\gamma - \chi}
\]
Here the dimensionless quantities $G_j$ are defined as follows
\[
G_c = \frac{C't_*^{3/2}}{w_*}, \quad G_e = \frac{E'w_*}{P_*t_*}, \quad G_m = \frac{w^2_*P_*t_*}{\mu't_*^2}, \quad G_v = \frac{Q_0t_*}{w_*t_*}, \quad G_k = \frac{K't_*^{1/2}}{E'w_*}
\]

**Leak-off-storage scaling:** If we impose the constraints $G_c = G_m = G_e = G_v = 1$, then we obtain four conditions to identify the characteristic quantities $t_*, t_*, p_*$, and $w_*$. The first of these constraints, equating the dimensionless leak-off coefficient to the dimensionless viscosity, identifies the time $t_*$ at which the transition from storage to leak-off dominated regimes occurs. The dimensionless toughness $G_k$ becomes a free parameter in this scaling.

**Viscosity-toughness scaling:** On the other hand, if we impose the constraints $G_k = G_m = G_e = G_v = 1$, then we obtain another set of characteristic quantities in which the $t_*$ represents the transition time from viscosity to toughness dominated regimes. The dimensionless leak-off coefficient $G_c$ becomes a free parameter in this scaling.

**Tip and Channel Regions:** Those elements containing the fracture tips are referred to the so-called tip region while those interior elements completely filled with fluid and not containing the tip form part of the so-called channel region. This logical decomposition of the problem is useful to identify the regions within the evolving fracture in which the appropriate tip asymptotic behavior is imposed as well as those computational mesh points that are used to locate the free boundary.

### 2.3 The discretized forward model

In this subsection, we describe the procedure used to discretize the governing equations as well as that required to determine the location of the fracture
front. This coupled numerical algorithm is a 1D version of the implicit level set algorithm [29] which was developed to locate the free boundary of a fracture by exploiting the tip behavior, which has been determined by asymptotic analysis (see [10, 11, 12, 13]). This algorithm forms the basis for the forward model used in the inversion experiments.

The region into which the evolving fracture is expected to propagate is discretized into \( N \) uniform elements of length \( \Delta \chi = 2a \). The field variables \( \Omega(\chi, \tau) \) and \( \Pi(\chi, \tau) \) are represented by their values at the centres of the elements. If a fracture tip does not coincide with the edge of an element a partially filled tip element is defined in which the width at the centre of the element represents the average volume of fluid in the tip. Thus it is possible to represent an evolving HF on a fixed Eulerian grid.

**The discrete elasticity equation**

We assume that \( \Omega(\chi, \tau) \) is approximated by a representation in terms of piecewise constant basis functions, i.e., \( \Omega(\chi, \tau) = \sum_m \Omega_m(\tau) H_m(\chi) \) where \( H_m(\chi) \) is the characteristic function for the \( m \)th element

\[
H_m(\chi) = \begin{cases} 
1 & \text{if } \chi \in (\chi_m - a, \chi_m + a) \\
0 & \text{if } \chi \notin (\chi_m - a, \chi_m + a)
\end{cases}
\]

We then substitute this approximation into the dimensionless elasticity equation (9), integrate the hypersingular kernel over each element, and collocate the equation at element centers. The integral equation is thereby reduced to the following system of linear equations relating the pressures \( \Pi_m \) and fracture widths \( \Omega_n \)

\[
\Pi_m(\tau) = \sum_n C_{m-n} \Omega_n(\tau)
\]

where

\[
C_m = \frac{G_e}{\pi \Delta \chi^{1/4}} \left( \frac{1}{4m^2 - 1} \right)
\]

**The discrete fluid-flow equation**

To discretize the lubrication equation we integrate the dimensionless lubrication equation (10) over the space-time interval \([\chi_m - a, \chi_m + a] \times [\tau - \Delta \tau, \tau]\). We then approximate the time integral of the flux gradient by the right hand rule to obtain a Backward Euler approximation and approximate the spatial integrals of \( \Omega \) by the midpoint rule to obtain the following discrete form of the fluid-flow equation:
\[
\Omega_m(\tau) = \Omega_m(\tau - \Delta \tau) + \Delta \tau A(\Omega) \Pi_f + \frac{L_m}{\Delta \chi} + \frac{\delta m_0}{\Delta \chi} \int_{\tau-\Delta \tau}^\tau \psi(\tau') d\tau' \tag{14}
\]

where \(A(\Omega)\) is the central difference operator defined by

\[
[A(\Omega)\Pi]_k = \frac{1}{\Delta \chi} \left( \Omega_{k+\frac{1}{2}}^3 \frac{\Pi_{k+1} - \Pi_k}{\Delta \chi} - \Omega_{k-\frac{1}{2}}^3 \frac{(\Pi_k - \Pi_{k-1})}{\Delta \chi} \right)
\]

and the leak-off term \(L_m\) is defined as

\[
L_m = 2 \int_{\chi_m-a}^{\chi_m+a} \left( \sqrt{\tau - \tau_0(\chi)} - \sqrt{\tau - \Delta \tau - \tau_0(\chi)} \right) d\chi
\]

**Locating the free boundary using tip asymptotics**

We now describe the procedure that can be used to locate the free boundary using the asymptotic expansion for the fracture width appropriate for the regime in which the fracture is propagating.

**Toughness-Storage Regime:** In this regime the asymptotic expansion for the width is of the the form

\[
\Omega \xi \sim 0 \sim \xi^{1/2} \tag{15}
\]

which can easily be inverted to yield

\[
\xi \sim \Omega^2 \tag{16}
\]

We observe that in this particular case the front location does not involve the velocity field. Now if we have a trial width \(\Omega\), which is sampled in the so-called channel region just next to the tip element, then (16) provides a good estimate of the distance between the collocation point at which \(\Omega\) is sampled and the actual tip position.

**Viscosity-Storage Regime:** In this regime the asymptotic expansion (7) for the width reduces to the form

\[
\Omega \xi \sim 0 \sim \beta v^{1/3} \xi^{2/3} \tag{17}
\]

where \(v\) is the normal velocity of the front.
Inverting this asymptotic relation we obtain

$$\xi \sim \left( \frac{\Omega}{\beta \nu^{1/3}} \right)^{\frac{2}{3}}$$

(18)

We observe that this asymptotic relation involves the normal velocity $v$ of the front. Determining the normal velocity by taking a divided difference approximation to the singular pressure gradient is undesirable as it involves an indeterminate limit. As an alternative, the local front velocity can be expressed in terms of two successive front locations $\xi_0$ and $\xi$

$$v = \frac{\xi - \xi_0}{\Delta \tau}$$

(19)

We substitute (19) into (18) to eliminate the velocity $v$ and rearrange terms to obtain the following cubic equation for the location $\xi$ of the front

$$\xi^3 - \xi_0 \xi^2 - \Delta \tau \left( \frac{\Omega}{\beta} \right)^3 = 0$$

(20)

3 Hydraulic Fracture Monitoring as a Non-stationary Inverse Problem

Data assimilation [19, 20] has been used in different fields either for calibrating models through parameter identification or for estimating states, or both. These formulations blend information provided by physical models with observed, often incomplete, noisy data obtained either from experiments or from field monitoring.

Traditional Data assimilation methods rely upon the Kalman Filter and extensions of this method. The main ideas, which are based on the use of Bayseian Inference, are described in detail in [21, 22]. The next section features a brief presentation of the basic equations and concepts. Indeed, Extended Kalman Filters (EKF) have been extensively employed for inverse problems related to a wide range of applications (see for example [23, 24, 25, 26]) and, particularly relevant for the present work, [27] in which the EKF was applied to crack detection.
3.1 A brief review of the EKF Method

An EKF estimates the state evolution of a physical system, within a probabilistic framework, through the systematic use of an evolution model and observed data, which is assumed to be corrupted by noise and can also be incomplete. Therefore, the goal, in the present context, is to obtain the state (here represented by discrete versions of $\Omega(\chi, \tau)$, $\Pi(\chi, \tau)$ and the unknown fracture domain $[\gamma^l, \gamma^r]$) at each instant of the fracture evolution with the help of deformation measurements and the discrete evolution model introduced in the previous section.

The filter is built upon a discrete state space representation of the system expressed in the following compact form:

\[
X_{k+1} = F_{k+1}(X_k, W_{k+1}) \\
Y_{k+1} = G_{k+1}(X_{k+1}, V_{k+1})
\]

Here $X_k = [\gamma^l, \gamma^r, \Omega(\chi, \tau), \Pi(\chi, \tau)]$ is a vector containing the states at instant $\tau_k$, $Y_k$ is the vector of observations provided by the tiltmeters, and $F$ and $G$ are the evolution and the observation models, respectively. The corresponding operators are nonlinear. Moreover $W$ and $V$ are stochastic variables introduced in order to account for possible uncertainties. The former is associated with the noise typically produced by measurements and the latter results from modeling errors that might be caused by space-time discretization of the continuous model or by un-modeled dynamics that might result from a failure to account for some significant physical process in the formulation of the idealized model. With the intent of producing a real-time monitoring scheme for the present application, the model $F$ is constructed using the numerical scheme described in section 2 to solve the coupled system of integro-partial differential equations (1)-(5).

The EKF comprises two stages, namely: prediction and update. The first stage corresponds to solving (21) on the time interval $[\tau_k, \tau_{k+1}]$. The explicit form of equation (21) is adopted here to follow the general notation often encountered in the literature, but it is worth mentioning that the fracture domain is only defined and computed implicitly through the model introduced previously. The second stage consists of inverting the observed data in order to obtain an estimation of the state vector at $\tau_{k+1}$.

Evolving these two stages in time corresponds to a sequential identification algorithm which is summarized below. The results of this algorithm are
the estimated values of the state vector at each instant and the corresponding measures of uncertainty associated with the estimation provided by the state covariance matrix $\Gamma$. In order to identify the values of the variables within the different stages of the algorithm we adopt a notation with a double subscript - reminiscent of that used to denote conditional probabilities. Thus, $X_{\tau_k|\tau_j}$ represents the estimation of the state at $\tau_k$ taking into consideration data observed at time $\tau_j$ ($j \leq k$).

- **Prediction:** given $X_{k|k}$ and $\Gamma_{k|k}$ compute

$$X_{k+1|k} = F_{k+1}(X_{k|k})$$  \hspace{1cm} (23)

$$\Gamma_{k+1|k} = DF_{k+1|k} \Gamma_{k|k} DF_{k+1|k}^T + \Gamma_{w_{k+1}}$$ \hspace{1cm} (24)

- **Measurement Update (data inversion):**

$$K_{k+1} = \Gamma_{k+1|k} J_{k+1}^T (J_{k+1} \Gamma_{k+1|k} J_{k+1}^T + \Gamma_{v_{k+1}})^{-1}$$ \hspace{1cm} (25)

$$X_{k+1|k+1} = X_{k+1|k} + K_{k+1}(Y_{k+1} - G(X_{k+1|k}))$$ \hspace{1cm} (26)

$$\Gamma_{k+1|k+1} = (I_d - K_{k+1} J_{k+1}) \Gamma_{k+1|k}$$ \hspace{1cm} (27)

where $DF_{k+1}$ is the Jacobian of $F$ computed at $X_{k|k}$, frequently referred to as the transition matrix, and $\Gamma_{w_{k+1}}$ is the covariance matrix associated with the noise process $W$. The matrix $K$ is the so called Kalman Gain and $J_{k+1}$ is the Jacobian of the observation operator $G$. $\Gamma_{v_{k+1}}$ is the covariance matrix corresponding to the noise in the measurements. The superscripts $T$ and $-1$ denote, respectively, transpose and inverse matrices.

The sequential identification algorithm described above requires knowledge of the initial state and its covariance. It also requires models for describing uncertainties to be attributed to the observation and evolution stages as well. These details will be addressed later.

### 3.2 An EKF applied to 1D-Hydraulic Fracture

We now combine the EKF algorithm described above with the numerical model outlined in section 2 to produce an algorithm to identify the state evolution of a hydraulic fracture which assumed to be propagating in a state of plane strain. In what follows we will refer to this algorithm as the EKFN algorithm in order to emphasize that the filter is built upon a numerical
model. The hydraulic fracture is constrained to grow along a line which is taken, without loss of generality, to be the horizontal axis. Since the source-point of the fracture, corresponding to the well-bore, is typically assumed to be known, the objective of the filter is to retrieve the fracture domain defined by the two extremities $\gamma^l$ and $\gamma^r$ over the time interval $[\tau_0, T]$. Moreover, the fracture opening, described by the function $\Omega(\chi, \tau)$ (with $\chi \in (\gamma^l, \gamma^r)$), also needs to be identified. The pressure distribution $\Pi(\chi, \tau)$, which is important in the design and evaluation of the stimulation, can be obtained from $\Omega(\chi, \tau)$, $\gamma^l$ and $\gamma^r$ by using the elasticity operator given in (9) or its discrete form (13). The identification process assumes that the following parameters are known: the injected flow rate $Q$, the elastic parameters of the surrounding medium $E$ and $\nu$, and the viscosity of the fracturing fluid $\mu$.

3.2.1 Evolution Model

The evolution model to be employed here consists of the discretized model equations described in section 2. The EKFN requires the solution of this nonlinear problem at each time-step. The state associated with the hydraulic fracture corresponds to the vector field $X = [\gamma^l, \gamma^r, \Omega]^T$, where $\Omega$ is the vector of fracture apertures sampled at the mesh points of the discrete model.

The last issue regarding the evolution model that needs to be addressed is the computation of the tangent map $DF$. The implicit nature of the algorithm adopted makes this quantity difficult to compute. In the discrete problem, $DF$ is an $N \times N$ matrix, with $N$ corresponding to the sum of two components defining the crack domain, plus two tip openings $\Omega^{tip}$, and $(N - 4)$ more channel crack openings $\Omega^c$. This matrix accounts for the way in which small perturbations in $X_k$, denoted as $\delta X_k$, are evolved in time by the algorithm. Here, $DF$, also called the sensitivity matrix, is replaced by an approximation built upon on the assumption that small perturbations in the crack domain, defined by $\gamma^{hr}$, have no direct influence on $\delta\Omega_{k+1}$. Therefore, the matrix $DF$ is organized in a structure of blocks in which the sub-matrix formed by the two first columns and the last $(N - 2)$ rows are filled with zeroes. Moreover, the relation between $\delta\Omega_{k+1}$ and $\delta\Omega_k$, expressed in the square sub-matrix formed by the last $(N - 2)$ rows and columns, is built upon the discrete version of the elasticity equation and the lubrication equation combined in the implicit time-stepping algorithm (eq.(61) in [29]). The remaining sub-matrix formed by the first two rows of $DF$ is obtained by differentiating the relation (17), assuming that the fracture is propagating in the viscosity
dominated regime. This approximation performed well in the inversions used to generate the numerical examples presented below.

### 3.2.2 Observation Model

The observation model uses the strain field generated by the fracture in the surrounding elastic medium. More specifically, the observation model to be used with the EKFN combines the Elasticity equation and measurements provided by tiltmeters [30]. These sensors, which are deployed either on the surface or along offset observation boreholes, measure the inclination induced by the induced strain field. The measured tilt angles associated with the horizontal \( (x) \) and vertical \( (y) \) directions defined relative to the reference plane, are given by:

\[
\omega_i = \omega(x_i, y_i) = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \quad i = 1, N sites
\]

where \((x_i, y_i)\) are the coordinates of the observation sites and \(u = (u_x, u_y)\) is the displacement field. Thus, the observation model is cast in the form of the following integral equation:

\[
\omega(x_i, y_i) = -\frac{2}{\pi} \int_{\ell_l}^{\ell_r} \frac{x'y_i \ w(x')}{{[(x_i - x')^2 + (y_i)^2]^2}} \ dx'
\]

\[
= -\frac{2}{\pi} G_s \int_{\gamma_l}^{\gamma_r} \frac{\chi'\zeta_i \ \Omega(\chi')}{{[(\chi_i - \chi')^2 + (\zeta_i)^2]^2}} \ d\chi'
\]

whith \(G_s = \frac{\omega}{\ell} \) and \(y = \ell_\ast \zeta\).

At this point, it is worth noticing that the above equation involves a non-linear relation between the crack boundary points \(\ell_l\) and \(\ell_r\) and the measured inclinations, which corresponds to a nonlinear elastic inverse problem. From that perspective, the EKFN could be interpreted as a regularization scheme which provides prior information embedded in the states furnished by the evolution model at each time instant.

The data inversion, which constitutes a second stage filter, requires the computation of \(J\), the Jacobian of the integral operator defined in (29) evaluated at the predicted state \(X_{k+1|k}\), which was computed in the first stage. The first step in obtaining \(J\) involves the directional derivative given by
\[ D_{\delta X_k \omega}(X_k) = \int_{\gamma^r} \tilde{G}(\chi, \chi') \delta \Omega_k d\chi' + \tilde{G}(\chi, \gamma^r_k) \Omega_k(\gamma^l_k) - \tilde{G}(\chi, \gamma^l_k) \Omega_k(\gamma^r_k) \]  

(30)

with \( \tilde{G} \) representing the operator introduced in the integrand of relation (29).

Due to the vanishing width boundary conditions (11), the last two terms of (30) are zero, which implies that the inclinations are not sensitive to first order perturbations associated with changes in the fracture domain. Therefore, updating \( \gamma^{l,r} \) does not follow the procedure employed by standard EKF implementations and two alternative approaches are presented below.

The main challenges related to the monitoring of hydraulic fractures by directly inverting data from tiltmeters using models of the form (29) have been addressed in detail in [30]. These authors analyze the limitations of such an inverse formulation, particularly when tiltmeters are placed far away from the hydraulic fracture itself - a situation which typically occurs whenever surface tiltmeters are to be used. Indeed, only global parameters such as the fracture orientation and volume can be obtained with any precision. Similar conclusions are drawn in [31] and [32] where only elastostatic data is used for the inversion. At this point, it is important to emphasize that the approach introduced here goes beyond inverting the sequence of quasi-static elastic snapshots because the data are coupled to an evolution model. This makes it possible to solve for the crack boundaries even though these terms are not explicitly present in the linearized equation (30). In order to achieve this we considered two alternatives.

**Modified EKFN:** In the first approach, the canonical EKFN algorithm is modified in that the update stage is split into two parts. The first stage comprises computing only the components of \( X_{k+1|k+1} \) corresponding to \( \Omega_{k+1|k+1} \) in (26) using the Jacobian defined in (30). Once the crack openings are updated, the algebraic equation defining the asymptoptic tip behavior (17) is inverted in order to obtain \( \gamma^l_{k+1|k+1} \) and \( \gamma^r_{k+1|k+1} \), in a similar way to that used by the forward algorithm [29].

**Canonical EKF with weak tip boundary conditions:** Since the implicit level set algorithm [29] is used to track the fracture evolution, the requirement of having to locate the boundary points of the domain within the update stage can be relaxed. This feature of the level set algorithm is due to the fact that the vanishing width boundary conditions (11) are only weakly enforced, so that the corresponding linearization of the discrete equations contains the
last two terms of (30) explicitly. Thus by combining the weak form of the boundary conditions in the discrete equations with the tip asymptotics (17) it possible to use the EKFN in its canonical form.

Both alternatives were tested and led to very similar results.

3.2.3 Summary of the EKFN algorithm

We briefly summarize the main steps of the proposed filter algorithm which starts from an initial state which is assumed to be known, and combines the readings from tiltmeters with the predictions provided by the implicit algorithm [29].

**EKFN algorithm**

- Initialize the state: \( k = 1, \quad X_1 \)
- Advance time step: \( k = k + 1, \quad \tau \leftarrow \tau + \Delta \tau \)
  - Solve the nonlinear forward problem for \( \gamma^{l,r} \) and \( \Omega \) using (13)-(14)
  - Given \( \gamma^{l,r} \) and \( \Omega \), compute \( DF \)
  - Update the covariance matrix \( \Gamma \) using (24)
  - Compute the Jacobian \( J \) using (30)
  - Compute the Kalman Gain matrix \( K \) using (25)
  - Update the state variables using (26)
  - Update the covariance matrix \( \Gamma \) using (27)
- end time step loop

4 Results

In this section we present results for three distinct examples which have been chosen to illustrate the efficacy as well as the shortcomings of the proposed monitoring technique. All the examples presented use noisy synthetic tiltmeter data produced by the numerical scheme presented in section 2. The typical scenario is depicted schematically in figure 2, where tiltmeters are placed either on the surface or along offset monitoring wells.
In order not to commit the so-called inverse crime [22], the simulated data has been computed subject to the following constraints: the data were obtained using a more dense mesh than the one used for the state estimation; for some of the situations, the time-step was also assumed to be different from that used to generate the synthetic data; zero-mean white noise was added to the tiltmeter outputs with standard deviation varying from 1% to 5% of the maximum value measured. This last condition implies that the matrix $V_k$ introduced in (22) is diagonal, having each non-zero entry defined as stated before. A typical time series of synthetic tiltmeter measurements with additional noise is shown in figure 3.

In initial verification tests, which are not presented here, the EKFN algorithm was able to faithfully reproduce both the fracture width and the tip locations for a HF propagating in a uniform \textit{in situ} confining stress field without leak-off. A significantly coarser mesh was used for the EKFN algorithm than that used to generate the synthetic data. These results are to be expected, since the forward model itself, even without feedback from the tilt measurements via the EKF, should provide some approximation to the HF used to generate the synthetic data.
In order to challenge the EKFN algorithm, we present results for three numerical experiments in which the model used to generate the synthetic data contains a significant parametric variation or a dominant physical process which is omitted from the forward numerical model used in the EKFN. In the first experiment, the confining stress field $\Sigma_0\phi(\chi)$ for the synthetic data is assumed to decrease linearly with increasing $\chi$, whereas the forward model used for the EKFN assumes a uniform confinement field. In the second experiment, the synthetic confinement field has two jump discontinuities while the EKFN confinement field is assumed to be uniform. In the third experiment, the synthetic data are generated by a model in which significant leak-off is present whereas the EKFN forward model assumes that the rock is impermeable. The first two examples may be classified as having parametric uncertainty, while the third example involves un-modeled dynamics, as a dominant physical process has been ignored in the forward model. In each case the covariance of the model and initial conditions were assumed to be diagonal matrices.

In all the experiments it was found that the location of the tiltmeters can have a significant impact of the efficacy of the method. As expected, the performance of the EKFN depends on the relative location of the advancing
fracture tips and the tiltmeter stations, on the number of measurement sites, and the extent to which they generate data which is independent. This situation is not static either, since, as the fracture evolves, the tilt array can move into and out of an advantageous position for measurement. This poses a challenge when deciding which results to present - do we use only the worst case results or the best results? In an attempt to provide a somewhat dispassionate assessment of the algorithm we have chosen to present results for arrays of 2, 3 or 5 tiltmeters to illustrate the limitations and the possibilities of the technique. In practice, the placement of the tiltmeters to provide resolution in a region of interest could, in itself, become the subject of simulation and optimization in the design of the deployment of the monitoring apparatus.

4.1 Fracture evolution in a linear in situ stress field

The synthetic data were generated assuming that \( \Sigma_0 \phi(\chi) = \alpha_0 - \alpha_1 \chi \), where \( \alpha_0 = 1 \) and \( \alpha_1 = 0.01 \). The synthetic model started at an initial fracture with a radius of \( \gamma^l = \gamma^r = 0.85 \) corresponding to a dimensionless time \( \tau = 1.63 \). The solution given by Carbonell [28] was used to initialize the numerical algorithm, which was assumed to propagate in a viscosity dominated regime. A mesh size of \( \Delta \chi = 0.1 \) and a time-step \( \Delta \tau = 0.0102 \) were used. The forward model used by the EKFN assumes a uniform in situ stress field \( \Sigma_0 \phi(\chi) = 1 \), uses a mesh size \( \Delta \chi = 0.2 \) and the same time-step, and is assumed to start from the same initial solution as the synthetic solution but sampled on the coarser grid.

Such a linear variation in the in situ stress field is typical in relatively homogeneous regions underground in which the increase in stress with depth is due to the overburden rock. The growth of a hydraulic fracture in such a stress field is asymmetric as it follows the trajectory of least resistance.

In figure 4 we plot the fracture lengths \( \gamma^l < 0 \) and \( \gamma^r > 0 \) for the synthetic model with a linear in situ stress field, the forward model with a uniform stress field, and the EKFN-uniform estimates using feedback from two tiltmeters located at \( (\chi_1, \zeta_1) = (0., 0.9238) \) and \( (\chi_2, \zeta_2) = (2., 0.9238) \). Even with this small number of tiltmeter stations the EKFN algorithm is able to detect and compensate for the asymmetry in the fracture growth. We observe that there is little change in the EKFN results if the time-step is refined. We also note that the left tip is estimated a little more accurately than the right tip, which is probably due to the distance between crack tip
and the sensor array.

Figure 4: Fracture tip positions $\gamma_l < 0$ and $\gamma_r > 0$ for linearly varying (solid line) and uniform (- - -) \textit{in situ} stress fields. The EKFN-uniform estimates with 2 tilt measurements are shown (- - - and . . . )

In figure 5 we plot the same data as in figure 4, except that the EKFN-uniform model uses feedback from an array of sensors comprising the same two sensors above augmented by an additional sensor located at $(\chi_3, \zeta_3) = (4., 0.9238)$. There is a significant improvement in the estimation of the fracture tip positions due to the additional data provided by the third tiltmeter.

In figure 6 we plot the same data as in figure 4 except that the EKFN
model now uses an array of five sensors constructed by adding two tiltmeters located at \((\chi_4, \zeta_4) = (6.0, 0.9238)\) and \((\chi_5, \zeta_5) = (8.0, 0.9238)\) to the 3-sensor array. In this case the additional tiltmeters yield an extremely accurate location of the right tip position, while there is a small improvement in the estimate of the left tip position.

In figures 7, 8, and 9 we compare synthetic, uniform, and EKFN-uniform estimates of the fracture openings \(\Omega\) at early \((\tau = 600)\), intermediate \((\tau = 25)\), and...
Figure 6: Fracture tip positions $\gamma^l < 0$ and $\gamma^r > 0$ for linearly varying (solid line) and uniform (dash-dot) *in situ* stress fields. The EKFN-uniform estimates with 5 tilt measurements are shown (dash-dot-dot), and advanced ($\tau = 1600$) sample times in the simulation. The EKFN solution using feedback from the 5 tiltmeter array provides a good estimate of the synthetic fracture opening throughout the simulation. The EKFN solution using 2 tilts is reasonably accurate initially, but exhibits significant spurious leak-off as the simulation progresses.
4.2 Fracture evolution in a discontinuous in situ stress field

For this example synthetic data were generated assuming that the in situ stress field has the following piecewise continuous behavior

\[
\Sigma_0 \phi(\chi) = \begin{cases} 
0.6 & \text{for } \chi < -3, \\
0.5 & \text{for } -3 < \chi < 3, \\
0.3 & \text{for } 3 < \chi 
\end{cases}
\]
Figure 8: Fracture openings $\Omega(\chi, \tau = 1000)$ for linearly varying (solid line) and uniform (- - -) in situ stress fields at an intermediate time. The corresponding EKFN estimates of the fracture opening using the 2 and 5 tiltmeter arrays are shown (-.-.-) and ( . . . ) respectively.

The well-bore is located at $\chi = 0$ so the fracture propagates symmetrically until the stress discontinuities are encountered at $|\chi| = 3$. Since the confining stress is larger for $\chi < -3.$ than it is for $\chi > 3.$, the fracture will propagate preferentially across the right-most stress jump rather than to the left. In fact, since the confining stress in the region $\chi > 3.$ is smaller than that in the interval $-3. < \chi < 3.$, the fracture will even tend to herniate into the region
Figure 9: Fracture openings $\Omega(\chi, \tau = 1600)$ for linearly varying (solid line) and uniform (- - -) *in situ* stress fields at an advanced time in the simulation. The corresponding EKFN estimates of the fracture opening using the 2 and 5 tiltmeter arrays are shown (-.-.-) and (. . .) respectively.

$\chi > 3$. Such piecewise constant stress fields are common underground due to the sedimentary deposition and genesis of the layered rock strata.

The synthetic model started with an initial fracture having a radius of $\gamma_l = \gamma_r = 2.75$ corresponding to a dimensionless time $\tau = 9.4924$. A mesh size of $\Delta \chi = 0.5$ and a time-step $\Delta \tau = 0.0949$ were used. The forward model used by the EKFN assumes a uniform *in situ* stress field $\Sigma_0 \phi(\chi) = 0.5$, a
mesh size $\Delta \chi = 0.5$, the same time-step, and is assumed to start from the same initial solution as the synthetic solution but sampled on the coarser grid.

In figure 10 we plot the fracture lengths $\gamma^l < 0$ and $\gamma^r > 0$ for the synthetic model with the discontinuous in situ stress field defined in (4.2), the forward model with a uniform stress field, and the EKFN-uniform estimates using feedback from two tiltmeters located at $(\chi_1, \zeta_1) = (0, 0.9238)$ and $(\chi_2, \zeta_2) = (2, 0.9238)$. The EKFN algorithm provides a reasonable location of the left tip while it produces a poor location of the right tip, which seems to track the right tip position associated with a uniform in situ stress field.

In figure 11 we plot the same data as in figure 10 except that the EKFN-uniform model uses feedback from three tiltmeters located at $(\chi_1, \zeta_1) = (0, 0.9328)$, $(\chi_2, \zeta_2) = (2, 0.9328)$ and $(\chi_3, \zeta_3) = (4, 0.9328)$. There is a significant improvement in the estimation of the right tip position due to the additional data provided by the third tiltmeter, while the location of the left tip position actually deteriorates. This somewhat surprising result is probably due to the spurious leak-off in the case of the 2-sensor array (see figure 13), which results in an estimation of the less rapidly advancing left tip which only seems to be more accurate.

By augmenting the sensor array with two additional tiltmeters located at $(\chi_4, \zeta_4) = (6, 0.9328)$ and $(\chi_5, \zeta_5) = (8, 0.9328)$, the location of the right tip is significantly improved see figure 12, while there is only a marginal improvement in the location of the left tip (compared to the 3-sensor array).

In figures 13, 14 and 15 we compare the fracture openings $\Omega$ of the synthetic, uniform, and EKFN-uniform estimates using 2, 3, and 5 tiltmeter arrays, respectively. In each figure the fracture openings $\Omega$ at an early $\tau = 23.73$ and later $\tau = 29.42$ time are plotted. The EKFN-uniform solution using feedback from the 5 and 3 tiltmeter arrays are clearly an improvement on the estimate given by the 2 tiltmeter array.

The above results demonstrate that the proposed method performs well even in the presence of significant amounts of noise. Indeed, similar results were obtained when the initial configuration provided to the filter were perturbed. Besides, a relatively small number of sensors were employed and were located sufficiently far from the fracture that the configuration could be interpreted as monitoring the fracture evolution from the surface.
Figure 10: Fracture tip positions $\gamma^l < 0$ and $\gamma^r > 0$ for discontinuous (solid line) and uniform (−−−) in situ stress fields. The EKFN-uniform estimates with 2 tilt measurements are shown (−−−).

4.3 Hydraulic Fracture Propagation with Leak-Off

In the previous two examples the HF were assumed to be propagating in impermeable media. In this example, the synthetic data is produced by the 1D numerical model in a permeable medium in which we assume that the dimensionless leak-off coefficient $G_c = 1$. The synthetic model started with an initial fracture having a radius of $\gamma^l = \gamma^r = 0.45$ corresponding to a dimensionless time $\tau = 0.688$. A mesh size of $\Delta \chi = 0.1$ and a time-step
Figure 11: Fracture tip positions $\gamma^l < 0$ and $\gamma^r > 0$ for discontinuous (solid line) and uniform ( - - - ) in situ stress fields. The EKFN-uniform estimates with 3 tilt measurements are shown ( - . - )

$\Delta\tau = 0.0079$ were used. The forward model used by the EKFN assumes no leak-off, a mesh size $\Delta\chi = 0.2$, the same time-step, and is assumed to start from the same initial solution as the synthetic solution but sampled on the coarser grid. Thus a significant component of the physical situation has been omitted from the forward model used by the EKFN.

In figure 16 we plot the fracture lengths $-\gamma^l$ and $\gamma^r$ for the synthetic model with leak-off, the forward model without leak-off, and the EKFN-
Figure 12: Fracture tip positions $\gamma^l < 0$ and $\gamma^r > 0$ for discontinuous (solid line) and uniform (- - -) in situ stress fields. The EKFN-uniform estimates with 5 tilt measurements are shown (- . - )

impermeable estimates using feedback from three tiltmeters located at $(\chi_1, \zeta_1) = (0., 0.9328)$, $(\chi_2, \zeta_2) = (2., 0.9328)$ and $(\chi_3, \zeta_3) = (4., 0.9328)$. The left and right fracture lengths for the synthetic permeable and the impermeable model have identical left and right fracture lengths due to the symmetry of the problem. The EKFN-impermeable estimates of the left an right tip positions are not identical because of the asymmetric location of the tiltmeter array relative to the well bore. We observe that the EKFN estimates of the tip
Figure 13: Fracture openings $\Omega$ for discontinuous (solid line) and uniform (- - - ) in situ stress fields at two sample times in the simulation. The corresponding EKFN estimates of the fracture openings using the 2 tiltmeter array are denoted by (-.-.- ).

positions are remarkably close to the actual locations used to generate the synthetic data.

In figure 17 we compare the fracture openings $\Omega$ of the synthetic permeable, impermeable, and EKFN-impermeable estimates using the three tiltmeter array. In each case, the fracture openings $\Omega$ are sampled at the time $\tau = 6.95$. The EKFN-impermeable fracture opening using feedback from the 3 tiltmeter array shows excellent agreement with that of the simulated data.
in which the medium was assumed to be permeable. Thus, in spite of the dominant un-modeled dynamics, which is the cause of the huge discrepancy between the permeable and impermeable $\Omega$ values, the feedback from the tiltmeter data via the EKFN is able to compensate and to provide excellent estimates of the fracture opening and tip positions.

Figure 14: Fracture openings $\Omega$ for discontinuous (solid line) and uniform (- - - ) \textit{in situ} stress fields at two sample times in the simulation. The corresponding EKFN estimates of the fracture openings using the 3 tiltmeter array are denoted by (---).
Figure 15: Fracture openings $\Omega$ for discontinuous (solid line) and uniform (---) in situ stress fields at two sample times in the simulation. The corresponding EKFN estimates of the fracture openings using the 5 tiltmeter array are denoted by (-.-.-).

5 Conclusions

The real-time monitoring of propagating HF is important for industrial applications in which it is desirable to avoid the penetration of the HF into environmentally sensitive regions, for example. Hitherto, the inversion of tiltmeter data has focussed on the inversion of elasto-static data sampled at snapshots taken during the well stimulation process. These measurements
Figure 16: Fracture tip positions $-\gamma^l$ and $\gamma^r$ for permeable (solid line) and impermeable (- - - ) media. The EKFN-impermeable estimates with 3 tilt measurements are shown (- . - and . . . )

were used to estimate the parameters in very simplified models of the fracture plane - such as an isolated displacement discontinuity or the identification of the moments of the fracture width. Because of the limited number of tiltmeters that can be deployed and the constraints on their location, there is little data for the purposes of inversion. This lack of information and the fact that the elasticity operator rapidly smooths the strain field with distance from the fracture, conspire to make the inverse problem ill-posed.
Figure 17: Fracture openings $\Omega$ for permeable (solid-red) and impermeable (- - - green) media at $\tau = 6.95$. The corresponding EKFN estimates of the left and right branches of the fracture opening using the 3 tiltmeter array is shown (-.-.-) and ( . . . ) respectively.

In this paper we have explored the possibility of connecting these isolated elasto-static measurements through a coupled elasto-lubrication forward model for the evolution of the HF itself. This approach means that all the causally admissible measurements can be deployed to determine the desired information about the evolving fracture geometry. We have explored the use of the EKF combined with a discrete coupled model based in the implicit level set algorithm in order to identify the fracture geometry. Data
from the tilt measurement time series is fed back to the forward model to provide corrections for parameter uncertainty or un-modeled dynamics.

Since we are using a forward model in the inversion process, there an expectation that the forward model itself might produce viable estimates of the fracture geometry. In the examples presented we have chosen to deliberately challenge the EKFN algorithm by ignoring significant parameter variations or physical processes that have a dominant effect on the fracture geometry. In spite of these hurdles, the EKFN was able to identify the fracture geometry with remarkable fidelity given the relatively few tiltmeter measurements that were used.

As is to be expected, we found that the number and location of the tiltmeter arrays can have a significant impact on the resolution of the EKFN. Although this is a limitation of the algorithm, it does open the possibility of using the algorithm itself to determine the optimal tiltmeter location before deployment in the field in order to achieve, for example, the best resolution in a particular region of interest.

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