Math 257/316 Assignment 1, 2015
Due Wednesday September 16 IN CLASS

Problem 1: (ODE Review) Find the general solutions of the following equations:

a. \( y' = (\sin x)y + 2x e^{-\cos x} \)
   \( y(0) = 1 \)
   Exact solution is: \( y(x) = e^{-\cos x} x^2 + \frac{1}{\cosh 1 - \sinh 1} e^{-\cos x} \)

b. \( x^2 y' = y \ln y - y' \)
   Exact solution is: \( y(x) = \exp(\exp(\arctan x + C_1)) \)

c. \( y'' - 3y' + 2y = 0 \)
   Exact solution is: \( y(x) = C_1 e^x + C_2 e^{2x} \)

\( y'' + 2y' + 2y = 0 \)

\( y(0) = 2 \)

\( y(x) = 5e^{-x} \sin x + 2e^{-x} \cos x \)

\( y(0) = 3 \)

e. \( y'' - 4y' + 4y = 0 \)
   Exact solution is: \( y(x) = C_1 e^{2x} + C_2 e^{2x} x \)

f. \( x^2 y'' + 4xy' + 2y = -1 - x \)
   Exact solution is: \( y(x) = -\frac{1}{2} - \frac{1}{6} x + \frac{C_1}{x} + \frac{C_2}{x^2} \)

g. \( x^2 y'' - xy' + y = 0 \)
   Exact solution is: \( y(x) = C_1 x + C_2 x \ln x \)

h. \( x^2 y'' + 2xy' + y = 0 \)
   Exact solution is: \( y(x) = \frac{C_1}{\sqrt{x}} \cos \left( \frac{1}{2} \sqrt{3} \ln x \right) + \frac{C_2}{\sqrt{x}} \sin \left( \frac{1}{2} \sqrt{3} \ln x \right) \)

Problem 2: (Power series solution warm-up): Consider the following first order linear ODEs:

\[ y' - (1 + 2x)y = 0 \] #
\[ xy' + (1 + x)y = 0 \] #

a. Solve the differential equations (1) and (2) using the appropriate integrating factors.

b. Expand the solution to (1) as Taylor series about the point \( x_0 = 0 \). Expand the exponential in the solution to (2) as Maclaurin series.

c. Now for (1) assume a power series solution of the form
   \[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]
   obtain a recursion for the coefficients \( a_n \). Use these recursions to determine the series representation of the solution. Compare this result to the series obtained in part b above.

d. Consider the following recursive strategy to generate an approximate solution to (1).

Rewrite (1) as
\[ xy' + y = -xy \]

Now assuming \( x \to 0 \) and discarding the right hand side of (1), find a first order approximation \( y_0 \) as the solution to
\[ xy'_0 + y_0 = 0 \]

Now substitute \( y_0 \) on the right side of (1) and solve for \( y_1 \)
\[ xy'_1 + y_1 = -xy_0 \]

Continue this process till you obtain \( y_2 \). How does \( y_2 \) compare with the series solution to (1) obtained in b? Can you use this series to motivate a modification to the series expansion (1) that would be appropriate to use to obtain a series solution to (1)?
Q 2(a) \[ \frac{dy}{y} = (1+2x) \, dx + c \Rightarrow \ln y = x + x^2 + c \Rightarrow y = A e^{x} e^{x^2} \]

\[ \frac{dy}{y} = x \, \frac{dy}{1} + (1+x) \, dx = 0 \]

\[ \int \frac{dy}{y} = \int \frac{x \, dx + c}{x(1+x)} = \ln y = \ln x^{-1} - x + c \Rightarrow y = A e^{-x} x \]

2(b) i) \[ y = A e^{x} e^{x^2} = A \left[ 1 + x + \frac{3}{2} x^2 + \frac{7}{6} x^3 + \frac{25}{24} x^4 + \ldots \right] \]

ii) \[ y = A e^{-x} x = \frac{A}{x} \left[ 1 - x + x^2 - \frac{x^3}{2} + \frac{x^4}{3!} - \frac{x^5}{4!} + \ldots \right] \]

2(c) \[ \ln y = \frac{y^1}{(1+2x)} y = 0 \]

\[ \frac{dy}{y} = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} \]

\[ \frac{dy}{y} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n - \sum_{m=1}^{\infty} 2a_{m-1} x^{m+1} \]

\[ = \sum_{m=0}^{\infty} a_{m+1} (m+1) x^m - \sum_{m=1}^{\infty} 2a_{m-1} x^m \]

\[ = (a_1 - a_0) x^6 + \sum_{m=1}^{\infty} \left[ a_{m+1} (m+1) - 2a_{m-1} \right] x^m = 0 \]

\[ x^0 \]

\[ a_1 = a_0 \]

\[ x^m \quad m \geq 1 \] \[ a_{m+1} = \frac{a_m + 2a_{m-1}}{m+1} \]

\[ m=1: \quad a_2 = \frac{(a_1 + 2a_0)}{2} = \frac{(a_0 + 2a_0)}{2} = 3a_0 / 2 \]

\[ m=2: \quad a_3 = \frac{(a_2 + 2a_1)}{3} = \frac{(3a_0 / 2 + 2a_0)}{3} = 7a_0 / 6 \]

\[ \therefore \quad y(x) = a_0 \left[ 1 + x + \frac{3}{2} x^2 + \frac{7}{6} x^3 + \ldots \right] \]
\[ d) \quad L_0 y = x y' + y + x y = 0 \]

Let \( L_0 y = x y' + y \)

Then \( L_0 y_0 = x y_0' + y_0 = (x y_0)' = 0 \rightarrow y_0 = \frac{A_0}{x} \quad A_0 = \text{const} \)

**STEP 2:**

\[ L_0 y_1 = x y_1' + y_1 = (x y_1)' = -x y_0 = -A_0 \]
\[ x y_1' = -A_0 \quad \rightarrow \quad y_1 = -A_0 x + A_1 \]

\[ L_0 y_2 = (x y_2)' = -x y_1 = -x (-A_0 + A_1) = A_0 x - A_1 \]

\[ x y_2 = A_0 x^2 - A_1 x + A_2 \]

\[ y_2 = \frac{A_2 x^2}{2} - A_1 x + \frac{A_2}{x} \]

Since this is a 1st order ODE, there is only one arbitrary constant. So the arbitrary constants introduced at each iteration must be the same i.e. \( A_0 = A_1 = A_2 \)

\[ y_2(x) = A_0 \left[ \frac{1}{x} - x + \frac{x^2}{2} \right] \]

\( y_2 \) agrees with the first two terms in the series expansion for the solution \( \frac{A_0 e^{-x}}{x} \)

* We could try to look for a series of the form \( y(x) = x^{-1} \left[ a_0 + a_1 x + a_2 x^2 + \ldots \right] \)
* Or more generally
  \[ y(x) = x^{-1} \left[ a_0 + a_1 x + a_2 x^2 + \ldots \right] \]