VARATIONAL METHODS

**E61:** \[ a \mathbf{x} + \mathbf{b} \quad \mathbf{x} = \mathbf{b}/a \]

Equivalent minimization problem

\[ E(x) = \frac{1}{2} a x^2 + b x \]

\[ 0 = \frac{dE}{dx} = ax + b \]

**E62:** Assume \( A \) is symmetric

\[ \mathbf{A} \mathbf{x} = \mathbf{b} \]

\[ E(x) = \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} - \mathbf{x}^T \mathbf{b} \]

\[ 0 = \frac{\partial E}{\partial \mathbf{x}} = \mathbf{A} \mathbf{x} - \mathbf{b} = 0. \]

**E63:** Equilibrium of a mass spring system

\[ \mathbf{p} \mathbf{\varepsilon} = \mathbf{\varepsilon} = \sum_{i=1}^{N} \frac{1}{2} k (u_i^m - u_{i-1}^m)^2 + f_i \mathbf{u}_i \]

\[ 0 = \frac{\partial E}{\partial \mathbf{\varepsilon}_m} = k (u_m^m - u_{m-1}^m) - k (u_{m+1}^m - u_m^m) + f_i \]

\[ = -k \left[ u_{m+1}^m - 2u_m^m + u_{m-1}^m \right] + f_i = 0 \]

\[ \left( \frac{k \Delta x}{A} \right) \mathbf{u}'' = \mathbf{f}_i \]

\[ F = k \mathbf{u} \]

\[ \sigma = \frac{F}{A} = \frac{k \Delta x}{A} \frac{d \mathbf{u}}{dx} \]

\[ E \mathbf{u}'' = 0 \] Equilibrium eq

\[ \sigma = E \mathbf{u}_x \]
Equilibrium of a Bar

\[ \sigma(x+\Delta x)A - \sigma(x)A = fA \Delta x \]

Body Force Per Unit Volume

\[ \sigma(x+\Delta x) - \sigma(x) = \frac{f}{\Delta x} \]

\[ \frac{\partial \sigma}{\partial x} = f. \]

Hook's Law:

\[ \sigma = E \frac{\partial u}{\partial x} \]

\[ \varepsilon \frac{\partial u}{\partial x} = f. \hspace{1cm} E \text{ constant} \]

\[ (\varepsilon \frac{\partial u}{\partial x})_x = f \hspace{1cm} \varepsilon(x). \]

Potential Energy in the Bar:

\[ V[u] = \int_0^L \frac{1}{2} E(u')^2 + f(u) \, dx \]

\[ u(0) = 0 = u(L) \]

\[ V \text{ is a functional} \]

\[ V : H^1 \rightarrow \mathbb{R} \]

\[ H^1 = \{ u : \int (u'')^2 \, dx < \infty \} \]

How can we minimize this functional \( V[u] \) with respect to \( u \)?

- Euler introduced a wonderful device to reduce this functional minimization problem to a standard calculus problem.

- Assume \( u_0 \) is the minimizer that we want.

- Consider a 1-parameter family of functions defined by

\[ u(x; \varepsilon) = u_0(x) + \varepsilon \eta(x) \]

where \( \eta(0) = 0 = \eta(L) \) and

is sufficiently smooth for the integrals to exist but otherwise \( \eta \) is arbitrary.
\[ V[u(x; \varepsilon)] = V[u_0 + \varepsilon u] = \frac{\varepsilon}{2} \left( u_0^0 + \varepsilon u \right)^2 + f u_0 + \varepsilon u \] dx = V(\varepsilon) \]

where \( V(\varepsilon) \) is a function of the parameter \( \varepsilon \) rather than a functional \( V[u] \).

To locate the minimum we differentiate W.R.T. \( \varepsilon \):

\[ 0 = \frac{\partial V(\varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon = 0} = \left[ \frac{\partial E(u_0^0 + \varepsilon u^0)}{\partial \varepsilon} \right]_{\varepsilon = 0} \eta_x + f \eta_x \ dx \]

\[ = \int_0^l \left[ \frac{\partial E}{\partial \varepsilon}(u_0^0)^2 + f(x) \eta(x) \right] \ dx \]

\[ = \left[ E(u_0^0)^2 \right]_0^l - \int_0^l \eta \left( f - \frac{\partial E}{\partial \varepsilon}(u_0^0)^2 \right) \ dx. \]

Since \( \eta(0) = 0 = \eta(l) \) the boundary terms vanish.

And since \( \eta(x) \) is arbitrary, we may choose \( \eta = \left( (E u_0^0)' - f \right)^2 \) dx

So that

\[ \int_0^l \left( (E u_0^0)' - f \right)^2 \ dx = 0 \]

Thus the only possibility is that

\[ (E u_0^0)' = f \] (1)

• (1) is a necessary condition for a minimum, i.e., if \( u_0(x) \) is a minimizer of \( V[u] \) then \( u_0 \) must satisfy \( (E u_0)' = f \).

• (1) is known as the Euler-Lagrange equation associated with minimizing the functional \( V[u] \).

\[ V[u] = \frac{\varepsilon}{2} E u^2 + f u \ dx \]
* Does \( U_0 \) : \((E,u)\) = \( f \) really minimize \( V[u] \)?

Consider any function \( u(x) \) satisfying the BC \( u(0) = 0 = u(L) \):

\[
V[u] - V[u_0] = \int_0^L \frac{E}{2} \dddot{x}^2 - \frac{E}{2} \dddot{u}_0^2 + f(u - u_0) \, dx
\]

Now \( \frac{1}{2} (\dddot{u} - \dddot{u}_0)^2 = \frac{1}{2} \dddot{x}^2 - \frac{2}{2} \dddot{u}_0^2 + \dddot{u}_0^2 \)

\[
= \frac{1}{2} \dddot{x}^2 - \dddot{u}_0^2 (\dddot{u} - \dddot{u}_0) - \dddot{u}_0^2
\]

\[
\therefore \frac{1}{2} (\dddot{u} - \dddot{u}_0)^2 + \dddot{u}_0^2 (\dddot{u} - \dddot{u}_0) = \frac{1}{2} \dddot{x}^2 - \dddot{u}_0^2
\]

\[
\therefore V[u] - V[u_0] = \int_0^L \frac{E}{2} (\dddot{u} - \dddot{u}_0)^2 + E \dddot{u}_0^2 (\dddot{u} - \dddot{u}_0) + f(u - u_0) \, dx
\]

\[
= \int_0^L \frac{E}{2} (\dddot{u} - \dddot{u}_0)^2 - \left[ (\dddot{u}_0)^2 - \int_0^L (u - u_0) \, dx + E \dddot{u}_0 (\dddot{u} - \dddot{u}_0) \right] \, dx
\]

\[
= \int_0^L \frac{E}{2} (\dddot{u} - \dddot{u}_0)^2 \, dx \geq 0.
\]

* Why complicate the differential equation problem by converting it to a minimization problem?

- Useful for approximation → Rayleigh-Ritz method
- Notice that \( V[u] \) is defined on a broader class of functions than the ODE which requires \( u \) be twice differentiable.
Consider \( L u = u'' = f \)
\[ u(0) = \alpha \quad u(1) = \beta \]

**Variational Calculus, Consider Variations \( \delta u \), \( \delta u \).**

\[ \mathcal{I} [u] = \int_0^1 \frac{1}{2} u'^2 + fu \, dx \]

\[ 0 = \delta \mathcal{I} [u] = \int_0^1 u' \delta u' + f \delta u \, dx \quad \delta u(0) = 0 = \delta u(1) \]

\[ 0 = u' \delta u' + f (u'' - f) \delta u \, dx \]

\( \delta u \) Arbitrary \( \Rightarrow \) \( u'' = f \).

**Rayleigh-Ritz Approximation**

Look for an approximation of the form

\[ u(x) = \sum_{n=1}^{N} \alpha_n \psi_n(x) \]

\[ \mathcal{I}[u] = \int_0^1 \frac{1}{2} \left( \sum_{n=1}^{N} \alpha_n \psi_n' \right)^2 + f \left( \sum_{n=1}^{N} \alpha_n \psi_n \right) \, dx \]

\[ 0 = \frac{\partial \mathcal{I}}{\partial \alpha_m} = \int_0^1 \left( \sum_{n=1}^{N} \alpha_n \psi_n' \right) \psi_m' + f \psi_m \, dx \]

\[ = \sum_{n=1}^{N} \alpha_n \int_0^1 \psi_n' \psi_m' \, dx + \int_0^1 f \psi_m \psi_m(x) \, dx \]

\[ A \alpha = b \]

\[ A_{mn} = \int_0^1 \psi_n' \psi_m' \, dx = A_{mn} \quad b_m = -\int_0^1 f \psi_m \, dx \]
**Example:**

\[ u''(x) = -x^2 \] \[ u(0) = 0 = u(1) \]

\[ u_{ex}(x) = \frac{x (1-x^3)}{12} \]

**Use a Single Quadratic Lagrange Basis Function**

\[ u(x) = \alpha x (1-x) = 4 \alpha x - 4 \alpha x^2 \]

\[ u'(x) = 4 \alpha - 8 \alpha x = 4 \alpha (1-2x) = \alpha N_1'(x) \]

\[ I[\alpha] = \frac{1}{2} \int u''(x)^2 + \int u'(x) \, dx = -\frac{1}{8} \alpha + \frac{8}{3} \alpha x^2 \]

\[ A_{11} = \int_0^1 N_1(x) N_1(x) \, dx = \int_0^1 16 (1-2x)^2 \, dx = 16/3 \]

\[ b_1 = -\int_0^1 (-x^2) (4x - 4x^2) \, dx = +1/5 \]

\[ \alpha = \frac{1}{5} / (16/3) = \frac{3}{80} \]
NATURAL VS ESSENTIAL BOUNDARY CONDITIONS

**ESSENTIAL BOUNDARY CONDITIONS:**

Notice that the basis function \( N_i(x) = 4x(1-x) \)

had to satisfy both boundary conditions \( u(0) = 0 = u(1) \).

Typically these involve the solution \( u \) at the boundary points, not derivatives of \( u \). Boundary conditions that need to be imposed on the trial solution are called essential BC.

**NATURAL BOUNDARY CONDITIONS**

In the case of derivative BC, it is possible to build the boundary condition into the energy functional to be minimized. In this case all that is required is that the trial solution have sufficient freedom to be able to satisfy this boundary condition. Such a boundary condition is called a natural BC.

Consider \( u'' = f \)

\[ u(0) = \alpha \quad u'(1) = \beta \]

And consider the energy functional

\[ E[u] = \int_0^1 \left( \frac{1}{2} u'^2 + fu \right) \, dx - \beta u(1) \]

\[ 0 = SE = \int_0^1 u' su' + fu su \, dx - \beta su(1) \]

\[ = u'su - \int_0^1 su [u'' - f] \, dx - \beta su(1) \]

\[ = [u'(1) - \beta] su(1) - u'(0) su(0) - \int_0^1 su [u'' - f] \, dx \]

Since \( u(0) = \alpha \) is an essential BC, we require \( su(0) = 0 \). Since \( su \) is arbitrary we obtain the following necessary conditions

\[ u'' = f \quad \text{and} \quad u'(1) = \beta. \]
\[ u'' = -x = f \quad u''(x) = 2 + 7x - x^3 \]

\[ u(0) = 2 \quad u(1) = 3 \]

**Essential** \quad **Natural**

**Linear Basis Functions:**

\[ U(x) = 2N_1(x) + aN_2(x) = 2(1-x) + ax \]

\[ U'(x) = a - 2 \]

\[ E(a) = \int_0^1 \left[ \frac{1}{2} (u')^2 + f(x)u \right] dx - 3U(1) \]

\[ = -1 - \frac{10a}{3} + \frac{(-2+a)^2}{2} \]

\[ \frac{dE}{da} = -10 + (-2+a) = 0 \quad a = \frac{16}{3} \]

**Alternative:**

\[ E[a] = \int_0^1 \left[ \frac{1}{2} (2N_1(x) + aN_2(x))^2 + (-x)(2N_1(x) + aN_2(x)) \right] dx - 3a \]

\[ = 2 \int_0^1 N_1(x)N_2(x) dx + a \int_0^1 N_2^2(x) dx - \int_0^1 x^2 dx - 3 \]

\[ A_{12} = \int_0^1 (-1)(1) dx = -1 \quad A_{22} = \int_0^1 (1)(1) dx = 1 \]

\[ 0 = 2(-1) + a \cdot 1 - \frac{16}{3} - 3 \]

\[ a = \frac{16}{3} \]

\[ U(x) = 2(1-x) + \frac{16}{3}x \]
**How do we arrive at a cost functional?**

Consider \( u'' = f \)
\[
    u(0) = \alpha \quad u'(1) = \beta
\]

\[
0 = \int_0^1 \nu (u'' - f) \, dx
\]
\[
= \nu u' \big|_0^1 - \int_0^1 u' \nu' + f \nu \, dx
\]
\[
= \nu(1) \beta - \nu(0) u'(0) - \int_0^1 u' \nu' + f \nu \, dx
\]

\( \nu(0) = 0 \)

\[
0 = \int_0^1 u' \nu' + f \nu \, dx - \nu(1) \beta. \quad (*)
\]

**Now what functional is \( (*) \) the first variation of?**

\[
E[u] = \int_0^1 \frac{1}{2} (u')^2 + f u \, dx - u(1) \beta
\]
Robin Boundary Condition

\[ u'' = f \]
\[ u(0) = \alpha \quad u'(1) + \beta u = \gamma \]

Consider the energy functional

\[
\mathcal{E}[u] = \int_{0}^{1} \left( \frac{1}{2} (u')^2 + f u \, dx \right) + u(1) \left[ \frac{\beta u(1)}{2} - \gamma \right]
\]

\[
0 = \delta \mathcal{E}[u] = \int_{0}^{1} u' \delta u' \, dx + \int_{0}^{1} f u \delta u \, dx + \left[ \beta u(1) - \gamma \right] \delta u(1)
\]

\[
0 = u' \delta u \bigg|_{0}^{1} - \int_{0}^{1} u'' \delta u \, dx + \left[ \beta u(1) - \gamma \right] \delta u(1)
\]

Since \( u(0) = \alpha \) is an essential BC \( \delta u(0) = 0 \)

\[
0 = \left[ u'(1) + \beta u(1) - \gamma \right] \delta u(1) - \int_{0}^{1} u'' \delta u \, dx
\]

Since \( \delta u \) is arbitrary the necessary conditions for a minimum are that \( \mathcal{E} \) should satisfy

\[ u'' = f \]
\[ u'(1) + \beta u(1) = \gamma \].
Eigenvalue Problems:

\[-u'' = \lambda u, \quad u(0) = 0 = u(1)\]

\[\int_0^1 u'' u \, dx = \lambda \int_0^1 u^2 \, dx\]

\[\lambda \int_0^1 u^2 \, dx = u^2 \bigg|_0^1 + \int_0^1 (u')^2 \, dx\]

\[\therefore \lambda = \frac{\int_0^1 (u')^2 \, dx}{\int_0^1 u^2 \, dx}\]

\[\lambda_1 = \min I[u] = \min \left\{ \int_0^1 (u')^2 \, dx \right\}\]

\[u_0 = 0, \quad u_1 = 0, \quad \int_0^1 u^2 \, dx \leq 1\]

\[0 = \delta I = 2 \left\{ \int_0^1 u \delta u \, dx \right\} \int_0^1 u^2 \, dx - \int_0^1 (u')^2 \, dx \]

\[2 \int_0^1 u \delta u \, dx - \int_0^1 (u')^2 \, dx = 2 \left\{ \int_0^1 u \delta u - \lambda_1 \int_0^1 u \delta u \, dx \right\}\]

\[0 = u \delta u \bigg|_0^1 - \int_0^1 \left( u'' + \lambda_1 u \right) \delta u \, dx\]

\[\delta u \text{ arbitrary} \implies -u'' = \lambda_1 u.\]
THE EIGENVALUE PROBLEM AS A CONSTRAINED MINIMIZATION PROBLEM

Recall \( \lambda_1 = \min \int u'^2 \, dx = \min \int u^2 \, dx \)
\[ \text{subject to } \int u^2 \, dx = 1 \]
\[ u|_0 = 0, \quad u|_1 = 0 \]

Consider \( J[u] = \int (u')^2 \, dx + \mu \int u^2 \, dx \) \( \mu = \text{Lagrange multiplier} \).

\[ 0 = 8 \int u' u'' \, dx + 2 \mu \int u^2 \, dx \]

\[ 0 = u' u'' - \frac{1}{2} \left[ u'' - \mu u \right] \, dx \]

Since \( \delta u \) is arbitrary, we obtain the necessary condition

\[ -u'' = \mu u \]

\( \therefore -\mu \) is an eigenvalue \( -u'' \)

Assuming \( \int u'^2 \, dx = 1 \) and \( -u'' - \mu u = 1 \)

\[ -\mu = -\mu \int u'^2 \, dx = -\frac{1}{2} \int u'' u' \, dx = -u' u'' + \frac{1}{2} (u')^2 \, dx = \lambda_1 \]

1) THE LAGRANGE MULTIPLIER \( \mu = -\lambda_1 \)

2) THE MINIMIZER OF \( \int (u')^2 \, dx \) SUBJECT TO \( \int u^2 \, dx = 1 \)

IS AN EIGENFUNCTION OF \( Lu = -u'' \) WITH EIGENVALUE \( \lambda_1 \).
Example: Determine the lowest eigenvalue & eigenfunction of

\[ L\mathbf{u} = -\mathbf{u}'' = \lambda \mathbf{u} \]

Subject to: \( \mathbf{u}(0) = 0 = \mathbf{u}(1) \)

**Exact Solution:**

\[ \mathbf{u}'' + \beta^2 \mathbf{u} = 0 \quad \beta = \lambda \]

\[ \mathbf{u} = A \cos \beta x + B \sin \beta x \]

\( 0 = \mathbf{u}(0) = A \quad 0 = \mathbf{u}(1) = B \sin \beta \Rightarrow \beta n = n\pi \quad n = 1, 2, \ldots \)

The lowest eigenvalue is \( \lambda_1 = \beta_1^2 = \pi^2 = 9.869 \)

**Approximation:**

\[ \mathbf{u}(x) = a_4 x (1-x) = a_n^2 (x) \]

\[ \mathbf{u}' = 4a - 8ax \]

\[ \int_0^1 (\mathbf{u})^2 dx = \frac{16a^2}{3} = 30 = 10 \]

\[ \int_0^1 \mathbf{u}^2 dx = \frac{8a^2}{15} \]

**OR**

\[ \min_0^1 (\mathbf{u})^2 dx = \min_0^1 16a^2/3 \]

Subject to: \( \int_0^1 u^2 dx = 8a^2/15 = 1 \) \( \Rightarrow a = \sqrt{\frac{15}{8}} \)

\[ \lambda_1 = \frac{16a^2}{3} = \frac{16}{3} \left( \frac{\sqrt{15}}{8} \right)^2 = \frac{30}{3} = 10 \]

\[ \mathbf{u} = a_n^2 (x) = a_4 (x-x^2) \]

\[ \mathbf{u}' = 4a - 8ax \]

\[ \delta \mathbf{u} = b N_2(x) \]

\[ \delta \mathbf{u}' = b (4-8x) \]

\[ 16 b \int_0^1 (N_2)^2 dx = \lambda_1 b \int_0^1 N_2^2 dx \]

**OR:**

\[ 0 = \delta \mathbf{u} \Rightarrow \int_0^1 \mathbf{u}' \delta \mathbf{u} dx = \lambda_1 \int_0^1 \mathbf{u}' \delta \mathbf{u} dx \]

**Let**

\[ \mathbf{u} = a N_2(x) = a_4 (x-x^2) \]

\[ \mathbf{u}' = 4a - 8ax \]

\[ \delta \mathbf{u} = b N_2(x) \]

\[ \delta \mathbf{u}' = b (4-8x) \]

\[ 16 b \int_0^1 (N_2)^2 dx = \lambda_1 b \int_0^1 N_2^2 dx \]

**OR:**

\[ a \left( \frac{16 - \lambda_1}{\frac{15}{3}} \right) = 0 \Rightarrow \lambda_1 = \frac{30}{3} = 10 \]
THE METHOD OF WEIGHTED RESIDUES AND THE WEAK FORM

WHAT DO WE DO IF THE PROBLEM IS DISSIPATIVE
SO THAT IT IS NOT POSSIBLE TO CONSTRUCT A DUAL ENERGY FUNCTIONAL?

Eg: \( u^i - u^e = 0 \quad u(x) = \frac{\sinh (x-1)}{\sinh (c)} \)
\( u(0) = 1 \quad u'(1) = 0 \)

MORE GENERALLY, GIVEN A DIFFERENTIAL OPERATOR
\[ Lu = f \]
\( u(0) = \alpha \quad u'(1) = \beta \)

CONSIDER THE RESIDUAL
\[ R(u) = Lu - f \]
AND REQUIRE THAT
\[ \int_0^1 R(u) \cdot v \, dx = 0 \quad \text{for arbitrary} \ v \ \text{in some space of functions} \]

Eg: 1) \( v(x) = \delta(x-x_k) \quad k = 1, 2, \ldots, N \)
\[ R(u(x_k)) = 0 \quad \text{the collocation method} \]

2) \( v(x) = x^0 x^1 x^2 \ldots x^k \quad T(x) \)
\[ \int_0^1 R(u(x)) x^k \, dx = 0 \quad \text{the method of moments} \]
Consider the one dimensional Poisson problem:
\[-u'' = f \quad u'' + f = 0 \tag{5}\]
\[u(a) = G \quad u'(b) = H\]

Write:
\[\int_a^b \nabla \{u'' + f\} \, dx = 0 \quad \forall \nabla\]
\[\left. \nabla u' \right|_a^b - \int_a^b \nabla u' \cdot f \, dx = 0\]
\[\nabla u(b) - \nabla u(a) = \int_a^b u' \cdot f \, dx = 0\]

Find \(u \in H^1_a = \left\{ u : \int_a^b (u')^2 \, dx < \infty, \quad u(a) = G \right\}\) weak form such that:

\[a(u,v) = \int_a^b u'v' \, dx = \nabla u(b) \cdot v + \int_a^b f \cdot v \, dx\]

For all \(v \in H^1_0 = \left\{ v : \int_a^b (v')^2 \, dx < \infty, \quad v(a) = 0 \right\}\)

Note: We have just shown that if \(u\) satisfies (5) then \(u\) must also satisfy (W).

Claim (Converse) if \(u\) is twice differentiable and \(u\) satisfies (W) then \(u\) is a solution of (5).

Proof: Let \(u\) solve (W), then \[\int_a^b u'v' \, dx = \nabla u(b) \cdot v + \int_a^b f \cdot v \, dx\]

\[\int_a^b u''v' \, dx = \nabla u(b) \cdot v + \int_a^b f \cdot v \, dx\]

\[\int_a^b u''(b) - \nabla u(b) \cdot v - \int_a^b u''(a) - \nabla u(a) \cdot v = 0\]

For arbitrary \(v \in H^1_0\)

\[u'' + f = 0 \quad \text{and} \quad u'(b) = 4\]

And since \(u \in H^1_a \Rightarrow u(a) = G\) it follows that \(u\) satisfies (5).
FINITE DIMENSIONAL APPROXIMATION USING FINITE ELEMENTS

Let \( \mathcal{V}_h(x) = \mathcal{G} N_0(x) + \sum_{n=1}^{N} u_n N_n(x) = \mathcal{V}_h^h \subset H^1_0 \)

For the Galerkin approximation we assume the same basis functions for \( \mathcal{V}_h(x) \) excluding the basis function \( N_0(x) \) since \( \mathcal{V}(0) = 0 \) so that \( \mathcal{V}_h(x) \in \mathcal{V}_h^h \subset H^1_0 \).

Making use of the weak form we must determine \( \mathcal{V}_h^h(x) \in \mathcal{V}_h^h \subset H^1_0 \) such that

\[
a(h, \psi^h) = \int_a^b \left[ \mathcal{G} N_0^\prime + \sum_{n=1}^{N} u_n N_n^\prime(x) \right] \left[ \sum_{m=1}^{N} \varphi_m N_m(x) \right] dx
\]

\[
= \mathcal{V}_h(b) H + \sum_{m=1}^{N} \varphi_m \int_a^b f(x) N_m(x) dx
\]

\[
= \sum_{m=1}^{N} \varphi_m \left[ \frac{b}{N} \mathcal{G} N_0 N_m \right] + \sum_{m=1}^{N} \varphi_m \int_a^b f(x) N_m(x) dx - \mathcal{V}_m(b) H - \frac{b}{a} \int_a^b f(x) N_m(x) dx \mathcal{V}_m^\prime \delta = 0
\]

for \( m = 1, \ldots, N \).

\[
= \sum_{m=1}^{N} \mathcal{A}_{mn} \mathcal{V}_m \mathcal{V}_n = \mathcal{V}_m(b) H + \frac{b}{a} \int_a^b f(x) N_m(x) dx - \frac{b}{a} \mathcal{G} \int_a^b N_0^\prime N_m^\prime(x) dx = \mathcal{L}_m
\]

where \( \mathcal{A}_{mn} = \int_a^b N_n^\prime(x) N_m^\prime(x) dx \).

This is a linear system \( \mathcal{A} \mathcal{V} = \mathcal{L} \) for the nodal values \( \mathcal{V}_n \).
\[ A_{mn} = \int_a^b N_m(x) N_n(x) \, dx \]
\[ = \sum_{e=1}^E \int_{x_{e-1}}^{x_e} N_m(x) N_e(x) \, dx \]
\[ N_a = \frac{1}{2} (1 + \varepsilon_a \xi) \]
\[ N_1(\xi) = (1 - \xi)/2 \]
\[ N_2(\xi) = \frac{1}{2} (1 + \xi) \]
\[ x(\xi) = x_{e-1} N_1(\xi) + x_e N_2(\xi) = \left( \frac{x_{e-1} + x_e}{2} \right) + \left( \frac{x_e - x_{e-1}}{2} \right) \xi \]
\[ \frac{dx}{d\xi} = \frac{he}{2} \quad \frac{ds}{dx} = \frac{2}{he} \]

Define the element stiffness matrix \( A_{pq}^e = \int_{x_{e-1}}^{x_e} N_p(x) N_q(x) \, dx \)
\[ A_{pq}^e = \int_{x_{e-1}}^{x_e} N_p(x) N_q(x) \, dx = \int_{-1}^{1} \frac{dN_a(s)}{ds} \frac{ds}{dx} \frac{dN_b}{ds} \frac{ds}{dx} \, dx \, ds \]
\[ = \frac{2}{he} \int_{-1}^{1} \xi_a \xi_b \, ds = \frac{S_a S_b}{he} \]
\[ b_1 = N_1(b) H + \int_{x_0}^{x_1} f(x) N_1(x) \, dx - G \int_{x_0}^{x_1} N_0(x) N_1(x) \, dx \]
\[ b_m = N_m(b) H + \int_{x_{m-1}}^{x_m} f(x) N_m(x) \, dx - G \int_{x_{m-1}}^{x_m} N_0(x) N_m(x) \, dx \]
\[ b_N = N_N(b) H + \int_{x_{m-1}}^{x_N} f(x) N_N(x) \, dx \]
\[ A_{pq} = \frac{1}{h_c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \]

Assume \( h_c = h \) all \( h \).

\[
\begin{array}{cccc|cc}
 u_0 & u_1 & u_2 & u_3 & u_{N-1} & u_N \\
-1 & 1 & -1 & -1 & 1+1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 \\
\frac{1}{h} & = & & & & \\
1+1 & -1 & u_{N-1} & & & \\
-1 & 1 & u_N & & & \\
\end{array}
\]

\[ b_1 = -G \left( -\frac{1}{h} \right) + \int_{I_2} f_N(x) dx \]

\[ b_2 = \int_{I_2} f_N(x) dx \]

\[
\begin{array}{cccc|cc}
 u_1 & b_1 & \\
-1 & 2 & -1 & b_2 & \\
-1 & 2 & -1 & = h \\
\end{array}
\]

\[ H + \int_{I_{N-1}} f_N(x) dx \]

\[
\begin{array}{cccc|cc}
 u_1 & b_1 & \\
2 & -1 & \\
-1 & 2 & -1 & b_2 & \\
-2 & 1 & u_N & b_N & \\
\end{array}
\]
Example: $u'' = -x$  \[ u_0 = 2 + \frac{7}{2}x - \frac{x^3}{6} \]

$u(0) = 6 \quad u'(0) = 3 = 4$

$N = 1$:

$$A_0 \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + A_1 \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = 3 + \int_0^1 N_1(x) \, dx$$

$$-\frac{1}{h} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = 3 + \frac{x^3}{3} \bigg|_0^1$$

$h = 1 = 0.5 \quad u_1 = 5 + \frac{1}{3} = 16/3$

$$\frac{h}{2} u_1(x) = 2 N_1(x) + \frac{16}{3} N_2(x) = 2(1-x) + \frac{16}{3} x$$

$N = 2$:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2/(1/2) + \frac{1}{2} \int_0^{1/2} x(2x) \, dx + \int_0^{1/2} x(2x-1) \, dx \\ \frac{1}{2} \int_0^{1/2} x(2x-1) \, dx \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 + \frac{x^3}{3} - x^2 \bigg|_0^{1/2} \\ \frac{1}{2} \int_0^{1/2} x(2x-1) \, dx \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 + \frac{1}{12} + (1 - \frac{2}{3}) - (\frac{1}{4} - \frac{1}{12}) \\ 3 + \left(\frac{2}{3} - \frac{1}{3}\right) \end{bmatrix} = \begin{bmatrix} 4 + \frac{1}{4} \\ 3 + 5/24 \end{bmatrix}$$

$\left(1 + \frac{1}{2}\right) 2 u_1 = 7 + 5 + 6 = 7 + 11/24 \Rightarrow u_1 = \frac{7 + 11}{2} = \frac{58}{24}$

$\left(1 + \frac{3}{8}\right) 2 u_2 = 10 + \frac{1}{4} + 10 \Rightarrow u_2 = 5 + \frac{1}{8} + 5$