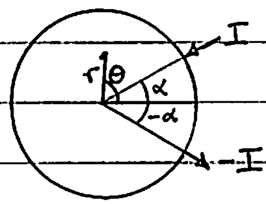
APPLICATION TO E.I.T.



$$\sigma \Delta u = 0$$

$$\sigma \frac{\partial u}{\partial n} = I\delta(\theta-\alpha) - I\delta(\theta+\alpha) = g(\theta)\sigma$$

ASSUMING $u(0)=0$

$$u(r,\theta) = -\frac{aI}{4\pi\sigma}\left[\ln\left\{\frac{[a^4+r^2a^2-2a^3r\cos(\theta-\alpha)][a^2+r^2-2ar\cos(\theta-\alpha)]}{r^2a^4}\right\}\right.$$

$$\left. - \ln\frac{[a^4+r^2a^2-2a^3r\cos(\theta+\alpha)][a^2+r^2-2ar\cos(\theta+\alpha)]}{r^2a^4}\right]$$

$$= \frac{aI}{2\pi\sigma} \ln\frac{[a^2+r^2-2ar\cos(\theta+\alpha)]}{[a^2+r^2-2ar\cos(\theta-\alpha)]}.$$

DIRECT SOLUTION:

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^n \sin n\theta \qquad \text{SINCE BC ARE ODD.}$$

$$g(\theta) = \left.\frac{\partial u(r,\theta)}{\partial r}\right|_{r=a} = \sum_{n=1}^{\infty} a_n n a^{n-1} \sin n\theta$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin m\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n n a^{n-1}\sin n\theta \sin m\theta\, d\theta = a_m m a^{m-1}\frac{\pi}{\pi}$$

$$\therefore \quad a_n = \frac{I}{\sigma n\pi a^{n-1}}[+\sin n\alpha - \sin(-n\alpha)] = \frac{+2I\sin n\alpha}{n\pi a^{n-1}}$$

$$\therefore \quad u(r,\theta) = \frac{+2aI}{\sigma\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{\sin n\alpha \sin n\theta}{n} \qquad C(A-B)-C(A+B) = 2\sin A\sin B$$

$$= +\frac{aI}{\sigma\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n[\cos n(\theta-\alpha) - \cos n(\theta+\alpha)]$$

$$= +\frac{aI}{\sigma\pi}\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{z_1^n}{n} - \frac{z_2^n}{n} \qquad z_1 = \left(\frac{r}{a}e^{i(\theta-\alpha)}\right)$$

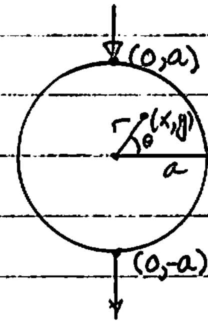
$$= -\frac{aI}{\sigma\pi}\,\mathrm{Re}\ln\frac{(1-z_1)}{(1-z_2)} \qquad z_2 = \left(\frac{r}{a}e^{i(\theta+\alpha)}\right)$$

$$= -\frac{aI}{2\sigma\pi}\ln\frac{|1-z_1|^2}{|1-z_2|^2} \qquad |1-\rho e^{i\Delta\theta}|^2 = (1-\rho e^{i\Delta\theta})(1-\rho e^{-i\Delta\theta})$$

$$= 1-2\rho\cos\Delta\theta+\rho^2$$

$$= \frac{aI}{2\sigma\pi}\ln\left[\frac{1-2\left(\frac{r}{a}\right)\cos(\theta+\alpha)+\left(\frac{r}{a}\right)^2}{1-2\left(\frac{r}{a}\right)\cos(\theta-\alpha)+\left(\frac{r}{a}\right)^2}\right]$$

SOLUTION FOR AN AXIAL PAIR ($\alpha = \pi/2$)



$$u(r,\theta) = \frac{aI}{2\pi\sigma} \ln \frac{[a^2+r^2+2ar\sin\theta]}{[a^2+r^2-2ar\sin\theta]}$$

EQUIPOTENTIAL LINES: $u = \text{CONST}$

$$\ln \frac{[x^2+(y+a)^2]}{[x^2+(y-a)^2]} = A$$

$$x^2+(y-a)^2 = r^2c^2 + r^2s^2 - 2ar\sin\theta + a^2 = a^2+r^2-2ar\sin\theta$$

$$x^2+(y+a)^2 = a^2+r^2+2ar\sin\theta$$

$$x^2+(y+a)^2 = \alpha(x^2+(y-a)^2) \quad ; \quad \alpha = e^A > 0$$

$$(1-\alpha)x^2 + (1-\alpha)y^2 + 2ay(1+\alpha) + (1-\alpha)a^2 = 0$$

$$(1-\alpha)\left\{x^2 + y^2 + 2ay\left(\frac{1+\alpha}{1-\alpha}\right)\right\} + (1-\alpha)a^2 = 0$$

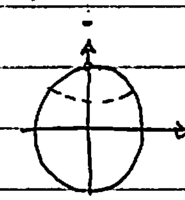
$$x^2 + (y+aK)^2 - K^2a^2 + a^2 = 0 \qquad K = \left(\frac{1+\alpha}{1-\alpha}\right)$$

$$a^2(1-K^2) = a^2\left(\frac{(1-\alpha)^2-(1+\alpha)^2}{(1-\alpha)^2}\right) = \frac{a^2(-4\alpha)}{(1-\alpha)^2}$$

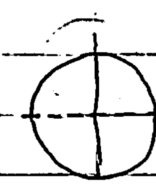
$$\therefore \quad x^2 + (y+aK)^2 = \frac{4a^2\alpha}{(1-\alpha)^2}$$

EQUIPOTENTIALS ARE CIRCLES WITH CENTRES ALONG $x=0$, RADIUS $2a\sqrt{\alpha}/|1-\alpha|$

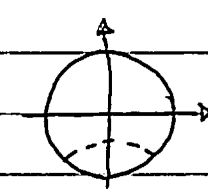
IF $\alpha = e^A > 1 \Rightarrow aK = a\left(\frac{1+\alpha}{1-\alpha}\right) < 0$ $\qquad x^2+(y-|aK|)^2 = \frac{4a^2\alpha}{(1-\alpha)^2}$



$\alpha = e^A = 1 \Rightarrow aK \to \infty$ STRAIGHT LINE

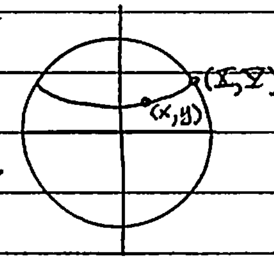


$\alpha = e^A < 1 \Rightarrow aK = a\left(\frac{1+\alpha}{1-\alpha}\right) > 0$



A. PEIRCE

INTERCEPTION POINTS BETWEEN LEVEL SETS & BOUNDARY



$$\frac{x^2+(y+a)^2}{x^2+(y-a)^2} = \alpha = \frac{\bar{X}^2+(\bar{Y}+a)^2}{\bar{X}^2+(\bar{Y}-a)^2}$$

WHERE THE POINT $(\bar{X},\bar{Y})$ FALLS ON THE CIRCLE & LEVEL SET:

SO THAT $\bar{X}^2+\bar{Y}^2=a^2$

$$\alpha = \frac{\bar{X}^2+(\bar{Y}^2+2a\bar{Y}+a^2)}{\bar{X}^2+(\bar{Y}^2-2a\bar{Y}+a^2)} = \frac{2a(a+\bar{Y})}{2a(a-\bar{Y})}$$

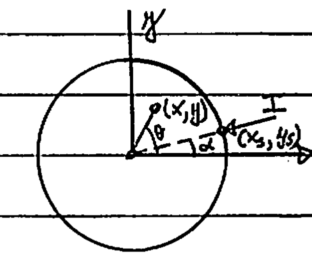
$$\therefore\ a+\bar{Y} = \alpha(a-\bar{Y}) \Rightarrow \bar{Y} = \frac{a(\alpha-1)}{(\alpha+1)}.$$

$$\therefore\ \bar{Y} = a\left\{\frac{\dfrac{x^2+(y+a)^2}{x^2+(y-a)^2}-1}{\dfrac{x^2+(y+a)^2}{x^2+(y-a)^2}+1}\right\}$$

$$= a\left\{\frac{x^2+y^2+2ay+a^2-x^2-y^2+2ay-a^2}{x^2+y^2+2ay+a^2+x^2+y^2-2ay+a^2}\right\}$$

$$\boxed{\bar{Y} = \frac{2a^2y}{(x^2+y^2+a^2)}, \qquad \bar{X} = \pm\sqrt{a^2-\bar{Y}^2}}$$

PROPERTIES OF GREEN'S FCN – EQUIPOTENTIAL LINES



$$u = \frac{aI}{2\sigma\pi} \ln\left[\frac{a^2+r^2-2ar\cos(\theta+\alpha)}{a^2+r^2-2ar\cos(\theta-\alpha)}\right]$$

POTENTIAL DUE TO A POINT SOURCE AT (x_s, y_s)

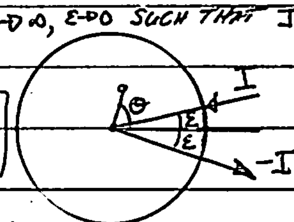
$$u = \frac{C}{4\pi} \ln\left[(x-x_s)^2+(y-y_s)^2\right]$$

$$(x-x_s)^2+(y-y_s)^2 = (r\cos\theta - a\cos\alpha)^2 + (r\sin\theta - a\sin\alpha)^2$$

$$= r^2\cos^2\theta - 2ar\cos\alpha\cos\theta + a^2\cos^2\alpha + r^2\sin^2\theta - 2ar\sin\theta\sin\alpha + a^2\sin^2\alpha$$

$$= a^2 + r^2 - 2ar\cos(\theta-\alpha)$$

SOLUTION FOR A DIPOLE STIMULATION · $J = I(2\varepsilon a)$ $\quad I \to \infty,\ \varepsilon \to 0$ SUCH THAT $J < \infty$



$$U = \lim_{\varepsilon\to 0} u_\varepsilon(r,\theta) = \left(\frac{aI}{2\sigma\pi}\right) \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon a} \ln\left[\frac{a^2+r^2-2ar\cos(\theta+\varepsilon)}{a^2+r^2-2ar\cos(\theta-\varepsilon)}\right]$$

$$= \frac{I}{4\sigma\pi} \lim_{\varepsilon\to 0}\left\{\frac{+2ar\sin(\theta+\varepsilon)}{[a^2+r^2-2ar\cos(\theta+\varepsilon)]} - \frac{-2ar\sin(\theta-\varepsilon)}{a^2+r^2-2ar\cos(\theta-\varepsilon)}\right\}$$

$$= \frac{I}{4\sigma\pi} \frac{4ar\sin\theta}{a^2+r^2-2ar\cos\theta}$$

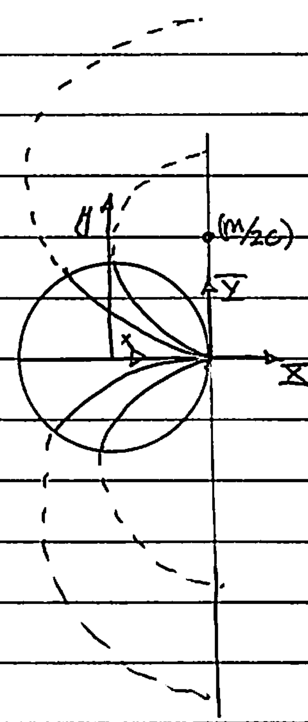
NOW $\quad (x-a)^2+y^2 = (r\cos\theta - a)^2 + r^2\sin^2\theta$

$$= r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta$$

$$= a^2 + r^2 - 2ar\cos\theta$$

$$\therefore \quad U = \frac{Ia}{\sigma\pi} \frac{y}{(x-a)^2+y^2}$$

EQUIPOTENTIAL LINES FOR A DIPOLE



CHANGE COORDINATES TO $(x-a) \to X,\ y \to Y$

$$m\frac{Y}{X^2+Y^2} = C \quad \text{FOR EQUIPOTENTIAL}$$

$$X^2+Y^2 = \frac{m}{C}Y^2$$

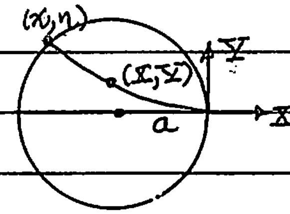
$$\therefore \quad X^2 + \left(Y - \frac{m}{2C}\right)^2 = \left(\frac{m}{2C}\right)^2$$

A. PEIRCE

INTERCEPTION POINT BETWEEN THE LEVEL SET AND THE CIRCLE.

$$\frac{m\,\bar{Y}}{\bar{X}^2+\bar{Y}^2} = \frac{m\,\eta}{x^2+\eta^2}$$ WHERE (x,η) FALLS ON THE CIRCLE BOUNDARY

SINCE (x,η) IS ON THE CIRCLE



$$(x+a)^2+\eta^2 = a^2$$

$$x^2+\eta^2 = -2xa$$

$$\therefore \frac{m\,\bar{Y}}{\bar{X}^2+\bar{Y}^2} = \frac{m\,\eta}{-2xa}$$

$$-2xa\bar{Y} = (\bar{X}^2+\bar{Y}^2)\eta$$

$$4a^2x^2\bar{Y}^2 = (\bar{X}^2+\bar{Y}^2)^2\eta^2$$

BUT $\eta^2 = -2xa - x^2$

$$\therefore\ 4a^2x^2\bar{Y}^2 + (2xa+x^2)(\bar{X}^2+\bar{Y}^2)^2 = 0.$$

$$x\left\{4a^2x\bar{Y}^2 + (2a+x)(\bar{X}^2+\bar{Y}^2)^2\right\} = 0$$

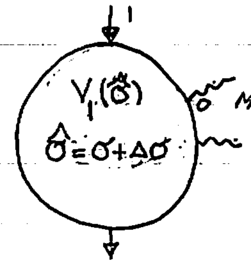
$$\therefore\ x\left\{x + \frac{2a(\bar{X}^2+\bar{Y}^2)^2}{4a^2\bar{Y}^2+(\bar{X}^2+\bar{Y}^2)^2}\right\} = 0.$$

$$x = 0 \quad \text{OR} \quad x = -\frac{2a(\bar{X}^2+\bar{Y}^2)^2}{4a^2\bar{Y}^2+(\bar{X}^2+\bar{Y}^2)^2}$$

A. PEIRCE

GENERALIZED SENSITIVITY THEOREM:

MEASUREMENT:



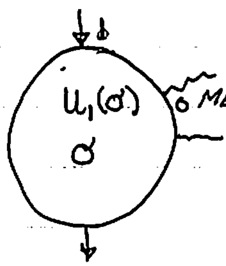
$$\nabla \cdot (\hat\sigma (\nabla V_1)) = 0$$

$$\text{BC 1: } \hat\sigma \frac{\partial V_1}{\partial n} = I(\delta_{\pi/2} - \delta_{-\pi/2}). \quad (BC_1).$$

Measure: Potential difference & calculate impedance

$$\hat Z_0 = (V_1(\pi/8) - V_1(0))/I = \Delta_0 V_1 / I$$

NOMINAL STATE:



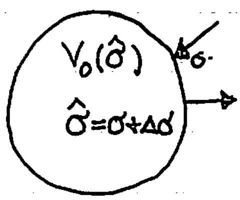
$$\nabla \cdot (\sigma \nabla U_1) = 0$$

$$\therefore \sigma \frac{\partial U_1}{\partial n} = I(\delta_{\pi/2} - \delta_{-\pi/2}) \quad (BC_1).$$

'MEASURE': Potential difference & calculate impedance

$$Z_0 = \{U_1(\pi/8) - U_1(0)\}/I = \Delta_0 U_1 / I.$$

RECIPROCITY STATE:



$$\nabla \cdot (\hat\sigma \nabla V_0) = 0$$

$$\hat\sigma \frac{\partial V_0}{\partial n} = I(\delta_{\pi/8} - \delta_0) \quad BC_0$$

$$0 = \int_\Omega V_0 \nabla(\sigma \nabla U_1)\, d\Omega = \int_{\partial\Omega} V_0 \sigma \frac{\partial U_1}{\partial n} ds - \int_\Omega \sigma \nabla V_0 \nabla U_1 \, d\Omega \quad (1)$$

$$0 = \int_\Omega U_1 \nabla(\hat\sigma \nabla V_0)\, d\Omega = \int_{\partial\Omega} U_1 \hat\sigma \frac{\partial V_0}{\partial n} ds - \int_\Omega \hat\sigma \nabla V_0 \nabla U_1 \, d\Omega. \quad (2)$$

$$(2)-(1) \Rightarrow \quad I(U_1(\pi/8) - U_1(0)) - I(V_0(\pi/2) - V_0(-\pi/2)) = \int_\Omega \overbrace{(\hat\sigma - \sigma)}^{\Delta\sigma} \nabla U_1 \nabla V_0 \, d\Omega. \quad (3)$$

Now

$$0 = \int_\Omega V_0 \nabla(\hat\sigma \nabla V_1)\, d\Omega = \int_{\partial\Omega} V_0 \hat\sigma \frac{\partial V_1}{\partial n} ds - \int_\Omega \hat\sigma \nabla V_1 \cdot \nabla V_0 \, d\Omega. \quad (4)$$

$$0 = \int_\Omega V_1 \nabla(\hat\sigma \nabla V_0)\, d\Omega = \int_{\partial\Omega} V_1 \hat\sigma \frac{\partial V_0}{\partial n} ds - \int_\Omega \hat\sigma \nabla V_1 \nabla V_0 \, d\Omega. \quad (5).$$

$$(5)-(4) \Rightarrow \quad \int_{\partial\Omega} V_1 \hat\sigma \frac{\partial V_0}{\partial n} ds = \int_{\partial\Omega} V_0 \hat\sigma \frac{\partial V_1}{\partial n} ds \Rightarrow I(V_1(\pi/8) - V_1(0)) = I(V_0(\pi/2) - V_0(0)) \quad (6).$$

Plug (6) into (3) to obtain:

$$(V_1(\pi/8) - V_1(0))/I - (U_0(\pi/8) - U_0(0))/I = -\int_\Omega \Delta\sigma \frac{\nabla U_1 \cdot \nabla V_0}{I^2} d\Omega$$

$$\boxed{\therefore \hat Z_0 - Z_0 = (\Delta_0 V_1 - \Delta_0 U_1)/I = -\int_\Omega \Delta\sigma \frac{\nabla U_1}{I} \cdot \frac{\nabla V_0}{I} d\Omega.}$$

NOTE: NO APPROXIMATION HAS BEEN MADE AT THIS STAGE.

Since we do not know $\hat{\sigma}$, ∇V_0 is impossible to calculate, so we assume $\hat{\sigma} = \sigma + \Delta\sigma$ where $|\Delta\sigma| << \sigma$.

$$V_0(\hat{\sigma}) = \underbrace{V_0(\sigma)}_{= \bar{u}_0} + \Delta\sigma \frac{\delta V_0}{\delta \sigma} + \cdots \qquad \frac{\delta V_0}{\delta \sigma} = \text{FRECHÉT DERIV.}$$

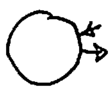
$$\therefore \quad V_0(\hat{\sigma}) \approx \bar{u}_0 + \Delta\sigma \frac{\delta \bar{u}_0}{\delta \sigma} + .$$

So the sensitivity theorem in this case is approximated by

$$\boxed{\hat{z}_0 - z_0 \approx \frac{-1}{I^2} \int_\Omega \Delta\sigma \, \nabla u_1 \, \nabla u_0 \, d\Omega} + O(\Delta\sigma^2)$$

BACKROJECTION: WITH AN INHOMOGENEOUS NOMINAL STATE:

Since we will be using the approximate sensitivity theorem consider

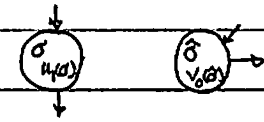


$$\nabla(\sigma(\nabla u_0)) = 0 \qquad \nabla(\sigma(\nabla u_1)) = 0$$

$$\sigma \frac{\partial u_0}{\partial n} = I(\delta_{\pi/8} - \delta_0) \qquad \sigma \frac{\partial u_1}{\partial n} = I(\delta_{\pi/2} - \delta_{-\pi/2})$$

ALGORITHM:

$$\hat{Z}_0 - Z_0 = \left(\frac{\Delta V_1}{I} - \frac{\Delta u_1}{I}\right) = -\int_\Omega \Delta\sigma \, \nabla u_1 \cdot \nabla V_0 \, d\Omega$$



$V_0 \sim u_0$

DISCRETIZE: BREAK Ω UP INTO ELEMENTAL REGIONS Ω_i ON EACH OF WHICH WE ASSUME $\Delta\sigma$ IS CONSTANT.

$$\Delta Z = \hat{Z} - Z = \left(\frac{\Delta V_1}{I} - \frac{\Delta u_1}{I}\right) \approx -\sum_{i=1}^{e} \Delta\sigma_i \int_{\Omega_i} \nabla u_1 \cdot \nabla u_0 \, d\Omega$$

UNDERDETERMINED SYSTEM

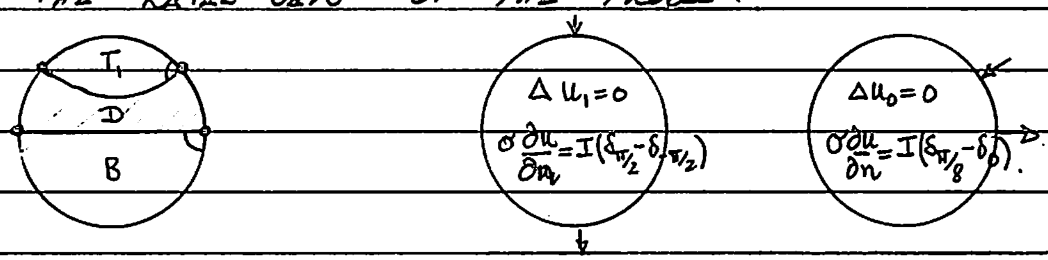
$$m\underbrace{\underset{\approx}{S}}_{N} \; \underset{\sim}{\Delta\sigma}^{N} = m \, \underset{\sim}{\Delta Z}$$

SOLVE BY LEAST SQUARES.

$$\underbrace{(\underset{\approx}{S}^T \underset{\approx}{S})}_{N\times N} \underset{\sim}{\Delta\sigma} = \underbrace{S^T \Delta \underset{\sim}{Z}}_{N}$$

BACK-PROJECTION

- VERY SPECIAL CASE IN WHICH THE SUBDOMAINS Ω_i ARE FORMED BY THE LEVEL-SETS OF THE PROBLEM



CLAIM 1: $\int_{T_i} \nabla u_1 \cdot \nabla u_0 \, d\Omega = 0 = \int_{B_i} \nabla u_1 \cdot \nabla u_0 \, d\Omega.$

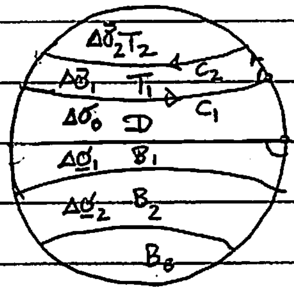
CLAIM 2: $\int_{D} \nabla u_1 \cdot \nabla u_0 \, d\Omega = \frac{I}{\sigma}\left[u_1(\pi/8) - u_1(0)\right] = \frac{I}{\sigma}\Delta u_1$

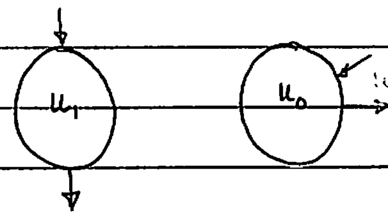
CLAIM 3: $\Delta Z = \frac{\Delta V_1}{I} - \frac{\Delta u_1}{I} = \frac{-1}{I^2}\int_\Omega \Delta\sigma \, \nabla u_1 \cdot \nabla u_0 \, d\Omega = \frac{-1}{I^2}\left\{\Delta\sigma_T \not\int_T + \Delta\sigma_D \int_D \nabla u_1 \cdot \nabla u_0 \, d\Omega + \Delta\sigma_B \not\int_B\right\}$

$$= -\frac{\Delta\sigma_D}{I^2}\left(\frac{I \, \Delta u_1}{\sigma}\right)$$

$$\therefore \quad \boxed{\Delta\sigma = -\sigma \frac{(\Delta V_1 - \Delta u_1)}{\Delta u_1}}$$

CLAIM 1: $\int_\Omega \Delta\sigma \, \nabla u_0 \nabla u_1 \, dV \simeq \Delta\sigma_0 \int_D \nabla u_0 \nabla u_1 \, dV.$





$$\int_{T_1} \Delta\sigma_1 \nabla u_1 \nabla u_0 \, dV = \Delta\sigma_1 \int_{T_1} \nabla u_1 \nabla u_0 \, dV \qquad \text{CONST ON ELEMENT}$$

$$= \Delta\sigma_1 \left\{ \int_{\partial T_1} u_1 \frac{\partial u_0}{\partial n} \, ds \right\}$$

$$= \Delta\sigma_1 \left\{ u_1(T/8) \int_{C_1} \frac{\partial u_0}{\partial n} ds + \int_{A} u_1 \frac{\partial u_0}{\partial n} ds \; (\to 0) + u_1(T/4) \int_{C_2} \frac{\partial u_0}{\partial n} ds \right\}$$

NOW $0 = \int_{T_1 \cup T_2} \nabla^2 u_0 \, dV = \int_{C_1} \frac{\partial u_0}{\partial n} ds + \int_{A} \frac{\partial u_0}{\partial n} ds \; (\to 0) \Rightarrow \int_{C_1} \frac{\partial u_0}{\partial n} ds = 0$

$0 = \int_{T_2} \nabla^2 u_0 \, dV = \int_{C_2} \frac{\partial u_0}{\partial n} ds + \int_{A} \frac{\partial u_0}{\partial n} ds \; (\to 0) \Rightarrow \int_{C_2} \frac{\partial u_0}{\partial n} ds = 0$

$\therefore \; \Delta\sigma_1 \int_{T_1} \nabla u_1 \nabla u_0 \, dV = 0$

$$\therefore \int_\Omega \Delta\sigma \, \nabla u_1 \nabla u_0 \, dV = \Delta\sigma_0 \int_D \nabla u_0 \nabla u_1 \, dV + \sum_{i=1}^{N} \Delta\sigma_i \int_{T_i} \nabla u_1 \nabla u_0 \, dV + \sum_{i=1}^{M} \Delta\sigma_i \int_{B_i} \nabla u_1 \nabla u_0 \, dV$$

$$= \Delta\sigma_0 \int_D \nabla u_0 \nabla u_1 \, dV.$$

CLAIM 2: $\int_D \nabla u_0 \cdot \nabla u_1 \, dV = \int_{\partial D} u_1 \frac{\partial u_0}{\partial n} \, ds$

BY PREVIOUS THM

$$= u_1(\pi/8) \int_{\partial S} \frac{\partial u_0}{\partial n} \, ds + \int_{\gamma} u_1 \frac{\partial u_0}{\partial n} \, ds \quad (= 0,\ = 0)$$

$$+ \, u_1(0) \int \frac{\partial u_0}{\partial n} \, ds + \int u_1 \frac{\partial u_0}{\partial n} \, ds$$

$$= \int u_1 \frac{I}{\sigma} [\delta_{\pi/8} - \delta_0] \, ds$$

$$= \frac{I}{\sigma} [u_1(\pi/8) - u_1(0)] = \frac{I}{\sigma} \Delta u_1$$

CLAIM 3: $\Delta Z = \frac{\Delta V_1}{I} - \frac{\Delta u_1}{I} = - \int_{\Omega} \Delta \sigma \frac{\nabla u_1 \cdot \nabla u_0}{I^2} \, d\Omega$

$$\simeq - \frac{\Delta \sigma_D}{I^2} (I \Delta u_1)/\sigma$$

$$= - \Delta \sigma_D (\Delta u_1 / I \sigma).$$

$$\therefore \boxed{\Delta \sigma_D = - \sigma \left(\frac{\Delta V_1 - \Delta u_1}{\Delta u_1} \right)}$$

THE BACK PROJECTION FORMULA