COMPOSITE GAUSS-LEGENDRE QUADRATURES

\[ I = \int_{a}^{b} f(x) \, dx = \sum_{k=0}^{N-1} x_k \int_{-1}^{1} f(\bar{x}_k + h_k s) \, ds \]

\[ = \sum_{k=0}^{N-1} h_k \int_{-1}^{1} \left( \frac{5}{6} f(\bar{x}_k - \frac{3}{5} h_k s) + \frac{8}{9} f(\bar{x}_k) + \frac{5}{9} f(\bar{x}_k + \frac{3}{5} h_k s) \right) \, ds \]

For a uniform mesh, pseudo-code:

\[ h = (b - a) / N; \quad h_0 = h / 2 \]

\[ x = a; h; b \]

\[ x_m = (x(1: end-1) + x(2: end)) \times 0.5 ; \]

\[ I = h \times \text{Sum} \left( \frac{5}{6} f(x_m - h_m \times \frac{3}{5}) + \frac{8}{9} f(x_m) + \frac{5}{9} f(x_m + h_m \times \frac{3}{5}) \right) \]

TRANSFORMATION

\[ x(s) = \left( \frac{x_k + x_{k+1}}{2} \right) + \left( \frac{x_{k+1} - x_k}{2} \right) s \]

\[ x(s) = \bar{x}_k + h_k s \Rightarrow dx = h_k \, ds \]

ADAPTIVE GAUSS INTEGRATION - GAUSS-KRONROD INTEGRATION - QUAD8K IN MATLAB

- We have seen the effectiveness of the adaptive transformation.

COMBINED WITH RICHARDSON EXTRAPOLATION TO YIELD ROMBERG INTEGRATION

TRAPEZOIDAL RICHARDSON SIMPSON RICHARDSON BOOLE'S

\[ O(h^2) \quad O(h^4) \quad O(h^4) \quad O(h^8) \]

- Problem with Gauss integration is that you have to throw away

THE FUNCTION EVALUATIONS IF YOU REFINE THE MESH

\[ \begin{array}{c|c|c|c|c|c} \hline 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \hline \end{array} \]

- In the 1960's, Kronrod devised a family of quadrature rules that use the N-point Gauss sample values and add another

\[ N+1 \] TO SAMPLE THE FUNCTION AT \[ 2N+1 \] AND IS EXACT FOR A POLY OF DEGREE \[ 3N+1 \].

Let \( x = \text{gauss points} \) AND \( \sigma = \text{the additional Kronrod points} \)

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} \hline 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \hline \end{array} \leftrightarrow K15 \]

\[ |E7 - K15| \sim \text{Error Estimate} \]
INTEGRATING A FUNCTION ON INFINITE INTERVALS

**Consider**

\[ I = \int_{a}^{\infty} f(x) \, dx \]

**Assume** \( |f(x)| \sim x^{-p} \) as \( x \to \infty \) where \( p > 1 \)

**Strategies:**
1. **Truncate the infinite interval**
   \[ I = \int_{a}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx = I_1 + I_2 \]
   **Discard**

   - **Use standard integration rules on** \( I_1 \)
   - **Try to get an error bound for** \( |I_2| \leq g(c) \leq \text{Tolerance} \)

   \[ I_2 = \int_{c}^{\infty} e^{-x} \cos x \, dx \]
   \[ \leq \int_{c}^{\infty} e^{-x} \, dx = e^{-c} \leq \text{Tolerance} = 10^{-6} \text{ say} \]
   \[ c \geq 6 \text{ ln} 10 \]

2. **Map the infinite interval to a finite one**
   
   **Choose** a transformation of the form
   \[ t = x^{1-p} \quad p > 1 \]
   **Then** \[ x = t^{1/(1-p)} \]
   \[ dx = \frac{1}{1-p} \, dt \quad x \to \infty \Rightarrow t \to 0 \]
   \[ I = \frac{1}{1-p} \int_{0}^{\infty} f \left( t^{1/(1-p)} \right) t^{-p} \, dt \]

   **We note that** as \( t \to 0 \)
   \[ f \left( t^{1/(1-p)} \right) t^{-p} \sim (t^{1-p})^{-p} t^{-p} \sim 1 \]

3. **Use specialized Gauss integration rules for infinite intervals**
   \[ \text{Eq. } \int_{0}^{\infty} e^{-x} f(x) \, dx = w_k f(s_k) \quad \text{where } w_k \text{ and } s_k \text{ are} \]
   **The Gauss-Legendre weights and abscissas since**
   \[ f(0, \infty) \text{ and } e^{-x} \text{ are the interval and weight function for the Legendre polynomials} \]