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SINGULAR INTEGRALS AND OPEN INTEGRATION FORMULAS

CONSIDER

$$I = \int_0^{\infty} \frac{e^x}{x^{1/2}} dx$$

PROBLEM AT $x=0$

1) SOMETIMES WE CAN TRANSFORM THE SINGULARITY AWAY $t=x^{1/2}$

$$x=t^2 \quad dx=2t dt \Rightarrow I = 2 \int_0^{\infty} e^{t^2} dt. \quad e^{t^2} \text{ HAS ALL ITS DERIVATIVES}$$

2) SUBTRACT THE SINGULARITY TO IMPROVE THE CONVERGENCE

$$\text{OBSERVE } e^x = 1 + x + x^2/2 + \dots \quad e^x - 1 = x + \frac{x^2}{2} + \dots$$

$$e^x - 1 - x = \frac{x^2}{2} + \dots$$

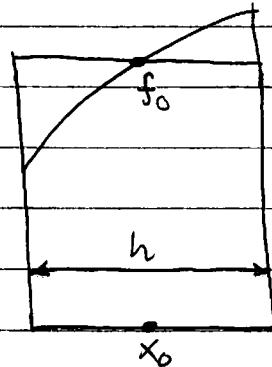
$$I = \int_0^{\infty} \frac{(e^x - 1) + 1}{x^{1/2}} dx = 2 + \int_0^{\infty} \left(\frac{e^x - 1}{x^{1/2}} \right) dx \quad f \sim x^{1/2} \quad f' \sim x^{-1/2}$$

$$= \int_0^{\infty} \frac{e^x - 1 - x}{x^{1/2}} + \frac{1+x}{x^{1/2}} dx = 2 + \frac{3}{2} + \int_0^{\infty} \left(\frac{e^x - 1 - x}{x^{1/2}} \right) dx \quad f \sim x^{3/2} \quad f' \sim x^{1/2}$$

3) USE AN OPEN INTEGRATION FORMULA

EG MIDPOINT RULE

$$\begin{aligned} \int_{x_0-h/2}^{x_0+h/2} f(x) dx &= \int_{x_0-h/2}^{x_0+h/2} f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2 f''(x_0)}{2!} + \dots dx \\ &= h f(x_0) + \frac{f''(s)}{2} \cdot \frac{h}{2} - \frac{1}{2} \left(\frac{h}{2} \right)^2 s^2 ds \end{aligned}$$



$$= h f(x_0) + \frac{h f''(s)}{2} \cdot \frac{h}{2} - \frac{1}{2} \left(\frac{h}{2} \right)^2 s^2 ds$$

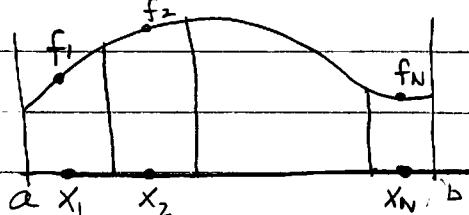
$$= h f(x_0) + \frac{f''(s)}{2} h^3 \quad \text{LOCAL TRUNCATION ERROR}$$

COMPOSITE MID POINT RULE

$$\int_a^b f(x) dx = h [f_1 + \dots + f_N] + \frac{h^2}{24} \sum_{k=1}^{N-1} f''(\xi_k) h$$

$$= M_N(h) + \frac{h^2}{24} \int_a^b f(s) ds$$

$$= M_N(h) + \frac{h^2}{24} [f'(b) - f'(a)]$$



GAUSS - INTEGRATION

ORTHOGONAL POLYNOMIALS

THERE EXIST FAMILIES OF POLYNOMIALS $\{\phi_n(x)\}$ EACH OF WHICH ARE ORTHOGONAL ON $[a, b]$ WITH RESPECT TO A WEIGHT FUNCTION $w(x)$ i.e.

$$\int_a^b w(x) \phi_m(x) \phi_n(x) dx = C_n S_{mn}$$

1. LEGENDRE POLYNOMIALS $[a, b] = [-1, 1]$ $w(x) = 1$

O.D.E. LEGENDRE EQ $Ly = (1-x^2)y'' - 2xy' + (n+1)ny = 0$

RECURSION: $P_n(x) = \frac{(2n-1)}{n} x \cdot P_{n-1}(x) - \frac{(n-1)}{n} P_{n-2}(x)$

$$P_0 = 1, P_1 = x, P_2 = \frac{3}{2} x \cdot x - \frac{1}{2} = \frac{1}{2} (3x^2 - 1), \dots$$

2. CHEBYSHEV POLYNOMIALS $[a, b] = [-1, 1]$ $w = \frac{1}{\sqrt{1-x^2}}$

$$Ly = (1-x^2)y'' - xy' + n^2 y = 0$$

$$\frac{1-x^2}{\sqrt{1-x^2}} y'' - \frac{x}{\sqrt{1-x^2}} y' + \frac{n^2}{\sqrt{1-x^2}} y = (\sqrt{1-x^2} y')' + \frac{n^2}{\sqrt{1-x^2}} y = 0$$

RECURSION: $T_n(x) = 2xT_{n-1} - T_{n-2}$

$$T_0 = 1, T_1 = x, T_2 = 2x^2 - 1, \dots$$

IMPORTANT FACT: THE POLYNOMIAL $\phi_n(x)$ IS ORTHOGONAL ON $[a, b]$ WITH RESPECT TO $w(x)$ TO ANY POLYNOMIAL $q_k(x)$ OF DEGREE $k < n$.

PROOF: SINCE $\{\phi_m(x)\}$ FORM A BASIS WE CAN WRITE $q_k(x)$ AS A LINEAR COMBINATION OF $\{\phi_m\}_{m=0}^k$

$$\text{i.e. } q_k(x) = \beta_0 \phi_0(x) + \beta_1 \phi_1(x) + \dots + \beta_k \phi_k(x) = \sum_{j=0}^k \beta_j \phi_j(x)$$

$$\text{Now } \int_a^b w(x) \phi_n(x) q_k(x) dx = \sum_{j=0}^k \beta_j \int_a^b w(x) \phi_n(x) \phi_j(x) dx = 0 \quad j=0, 1, \dots, k < n$$

QED

GAUSS-LEGENDRE INTEGRATION:

CONSIDER THE INTEGRATING A FUNCTION $f(x)$ OVER THE INTERVAL $[-1, 1]$ BY INTERPOLATING f BY A POLYNOMIAL $P_{2N-1}(x)$ OF DEGREE $2N-1$ AT THE $2N$ POINTS x_1, \dots, x_{2N} .

$$\int_{-1}^1 f(x) dx = \int_{-1}^1 P_{2N-1}(x) dx + \frac{f^{(2N)}(\xi)}{(2N)!} \int_{-1}^1 (x-x_1) \dots (x-x_{2N}) dx$$

$$= \int_{-1}^1 \sum_{k=1}^{2N} f_k l_k(x) dx + E_{2N-1}$$

$$= \sum_{k=1}^{2N} f_k w_k + E_{2N-1}$$

WHERE

$$w_k = \int_{-1}^1 l_k(x) dx = \frac{\int_{-1}^1 (x-x_1) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_{2N}) dx}{(x_k-x_1)(x_k-x_{k-1})(x_k-x_{k+1}) \dots (x_k-x_{2N})}$$

NOW FOR $k \geq N+1$

$$w_k = \frac{\int_{-1}^1 (x-x_1) \dots (x-x_N)(x-x_{N+1}) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_{2N}) dx}{\underbrace{\int_{-1}^1 (x-x_1) \dots (x-x_N)(x-x_{N+1}) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_{2N}) dx}_{Q_{N+1}(x)}}$$

IF x_1, \dots, x_N ARE $\rightarrow C P_N(x)$ THE ZEROS OF P_N

THUS IF WE CHOOSE x_1, \dots, x_N TO BE THE ZEROS OF THE N TH DEGREE LEGENDRE POLYNOMIAL $P_N(x)$ THEN $w_k = 0$ FOR $N+1 \leq k \leq 2N$.

IN THIS CASE

$$\int_{-1}^1 f(x) dx = \sum_{k=1}^N w_k f_k + E_{2N-1}$$

WILL INTEGRATE A POLYNOMIAL OF DEGREE $2N-1$ EXACTLY WITH ONLY N SAMPLES OF f .

NOTE: 1) THERE IS NO LOSS OF GENERALITY IN ASSUMING $\int_{-1}^1 f dx$ AS INTEGRALS ON MORE GENERAL INTERVALS $[a, b]$ CAN BE TRANSFORMED TO $[-1, 1]$ BY THE TRANSFORMATION $s = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)x$.

2) SINCE THE SAMPLE POINTS ARE NOT UNIFORM WE CONSIDER $h = \max_k |x_{k+1} - x_k|$. WE EXPECT THE LOCAL TRUNCATION ERROR TO BE $O(h^{2N+1})$ AND THE TRUNCATION ERROR OF THE COMPOSITE G-L RULE TO BE $O(h^{2N})$.

