

LECTURE 8

SIMPSON'S RULE

$$\int_{x_0}^{x_2} f(x) dx = \int_{x_0}^{x_2} p_2(x) dx + \frac{f(\xi)}{3!} \int_{x_0}^{x_2} (x-x_0)(x-x_1)(x-x_2) dx$$

$$\int_{x_0}^{x_2} p_2(x) dx = h \int_{-1}^1 f_0 \frac{s}{2}(s+1) + f_1(1-s^2) + f_2 \frac{s}{2}(s+1) ds$$

$$= h \left\{ f_0 \frac{\xi^3}{3} \Big|_0^1 + f_1 2 \left[\frac{s-s^3}{3} \right]_0^1 + f_2 \frac{2\xi^3}{3} \Big|_0^1 \right\}$$

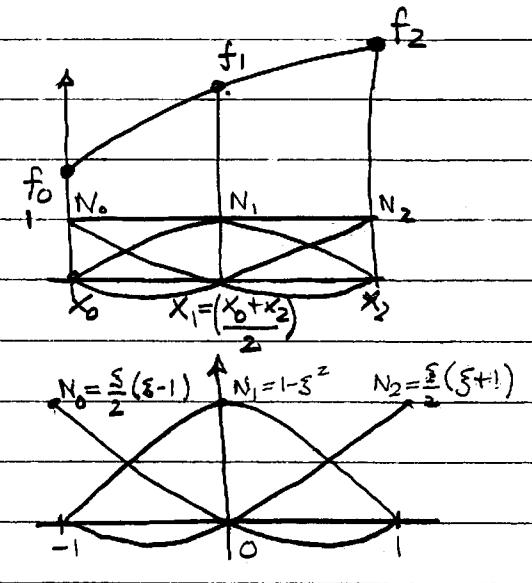
$$= \frac{h}{3} \{ f_0 + 4f_1 + f_2 \}$$

$$x-x_0 = h(1+\xi) \quad (x-x_1) = h\xi \quad x-x_2 = h(\xi-1)$$

$$E_2 = \frac{f(\xi)}{3!} \int_{x_0}^{x_2} (x-x_0)(x-x_1)(x-x_2) dx$$

$$= \frac{f(\xi)}{3!} h^4 \int_{-1}^1 (1+\xi) \xi (\xi-1) d\xi$$

$$= f(\xi) \frac{h^4}{3!} \int_{-1}^1 \xi^3 - \xi d\xi = 0$$



$$X = x_0 N_0(\xi) + x_1 N_1(\xi) + x_2 N_2(\xi)$$

$$= x_1 + \frac{(x_2-x_0)\xi + (x_0-2x_1+x_2)\xi^2}{2}$$

$$x = x_1 + h\xi \quad dx = h d\xi$$

$$E_4 = \frac{f^{(4)}(\xi)}{4!} h^5 \int_{-1}^1 \xi^4 - \xi^2 d\xi = \frac{f^{(4)}(\xi)}{4!} h^5 \left[\frac{\xi^5}{5} - \frac{\xi^3}{3} \right]_0^1 = - \frac{f^{(4)}(\xi) h^5}{3! 15} = - \frac{f^{(4)}(\xi) h^5}{90}$$

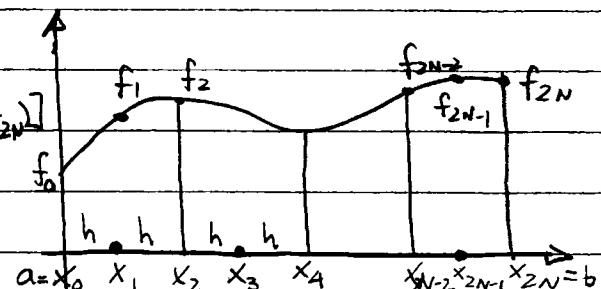
COMPOSITE SIMPSON'S RULE

$$\int_a^b f(x) dx = \frac{h}{3} [f_0 + 4f_1 + f_2] + [f_2 + 4f_3 + f_4] + \dots + [f_{2N-2} + 4f_{2N-1} + f_{2N}]$$

$$- \frac{h^4}{240} \sum_{k=1}^N f^{(4)}(\xi_k) (2h)$$

$$\approx S_{2N} - \frac{h^4}{180} \int_a^b f^{(4)}(s) ds$$

$$= S_{2N} - \frac{h^4}{180} [f^{(4)}(b) - f^{(4)}(a)]$$



COMPOSITE TRAPEZIUM RULE + RICHARDSON EXTRAPOLATION YIELDS SIMPSON'S RULE

$$\begin{array}{ccccc} f_0 & h & f_1 & f_2 & \\ \cdot & \cdot & \cdot & \cdot & \\ f_0 & 2h & f_2 & & \end{array}$$

$$\begin{array}{ccccc} f_{2N-2} & f_{2N-1} & f_{2N} & & \\ \cdot & \cdot & \cdot & \cdot & \\ f_{2N-2} & & f_{2N} & & \end{array}$$

$$T(h) = \frac{h}{2} [f_0 + 2f_1 + 2f_2 + \dots + 2f_{2N-2} + 2f_{2N-1} + f_{2N}]$$

$$T(2h) = h [f_0 + 2f_2 + \dots + 2f_{2N-2} + f_{2N}]$$

$$I(0) = \frac{4T(h) - T(2h)}{3} + O(h^4)$$

$$= 2A \frac{h}{3} [f_0 + 2f_1 + 2f_2 + \dots + 2f_{2N-2} + 2f_{2N-1} + f_{2N}]$$

$$- \frac{h}{3} [f_0 + 2f_2 + \dots + 2f_{2N-2} + f_{2N}] + O(h^4)$$

$$= \frac{h}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + \dots + 2f_{2N-2} + 4f_{2N-1} + f_N] + O(h^4)$$

$$= S_{2N}(h) + O(h^4)$$

ADAPTIVE SIMPSON INTEGRATION OF $I[a, b] = \int_a^b f(x) dx$

CONSIDER TWO APPROXIMATIONS OF THE CONTRIBUTIONS TO $I[a, b]$ ON THE SUBINTERVAL $I[x_0, x_4]$ THAT USE SIMPSON'S RULE:

$$S_2 = \frac{h}{3} [f_0 + 4f_2 + f_4] - \frac{f^{(4)}(\xi)}{90} h^5$$

$$\begin{array}{ccccccc} f_0 & f_1 & f_2 & f_3 & f_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$

$$S_4 = \frac{h/2}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + f_4] - \frac{h^5}{270} [f^{(4)}(\xi_1) + f^{(4)}(\xi_2)].$$

NOW ASSUME $f^{(4)} \sim C$ A CONSTANT, THEN

$$S_2 = I[x_0, x_4] - Ch^5/90 \quad \text{AND} \quad S_4 = I[x_0, x_4] - Ch^5/90 \cdot \frac{1}{16}$$

$$\therefore S_4 - S_2 = \frac{Ch^5}{90} \left(\frac{15}{16} \right)$$

NOW SINCE $|S_4 - I[x_0, x_4]| \leq |C| h^5 = \frac{1}{15} |S_4 - S_2|$ IS AN ESTIMATE OF THE ERROR

IF $\frac{1}{15} |S_4 - S_2| < \text{TOL}_1 S_4$ THEN PROCEEDED WITH THE CALCULATION OF $I[x_4, x_8]$

IF $\frac{1}{15} |S_4 - S_2| > \text{TOL}_1 S_4$ THEN REFINING THE MESH.

SINGULAR INTEGRALS AND OPEN INTEGRATION FORMULAS

CONSIDER

$$I = \int_0^{\infty} \frac{e^x}{x^{1/2}} dx$$

PROBLEM AT $x=0$

1) SOMETIMES WE CAN TRANSFORM THE SINGULARITY AWAY $t=x^{1/2}$

$$x=t^2 \quad dx=2tdt \Rightarrow I = 2 \int_0^{\infty} e^{t^2} dt. \quad e^{t^2} \text{ HAS ALL ITS DERIVATIVES}$$

2) SUBTRACT THE SINGULARITY TO IMPROVE THE CONVERGENCE

$$\text{OBSERVE } e^x = 1 + x + x^2/2 + \dots \quad e^x - 1 = x + \frac{x^2}{2} + \dots$$

$$e^x - 1 - x = \frac{x^2}{2} + \dots$$

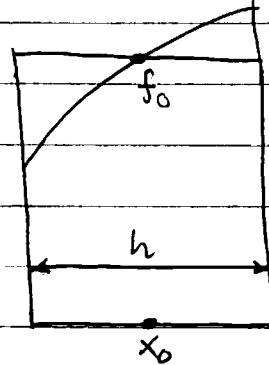
$$I = \int_0^{\infty} \left(\frac{e^x - 1}{x^{1/2}} + \frac{1}{x^{1/2}} \right) dx = 2 + \int_0^{\infty} \left(\frac{e^x - 1}{x^{1/2}} \right) dx \quad f \sim x^{1/2} \quad f' \sim \frac{x^{-1/2}}{2}$$

$$= \int_0^{\infty} \frac{e^x - 1 - x}{x^{1/2}} + \frac{1+x}{x^{1/2}} dx = 2 + \frac{3}{2} + \int_0^{\infty} \left(\frac{e^x - 1 - x}{x^{1/2}} \right) dx \quad f \sim x^{3/2} \quad f' \sim x^{1/2}$$

3) USE AN OPEN INTEGRATION FORMULA

EG MIDPOINT RULE

$$\begin{aligned} & \int_{x_0-h/2}^{x_0+h/2} f(x) dx = \int_{x_0-h/2}^{x_0+h/2} f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2 f''(x_0)}{2!} + \dots dx \\ &= h f(x_0) + \frac{f''(s)}{2} \cdot \frac{h}{2} \int_{x_0}^{x_0+h} \left(\frac{h}{2} \right)^2 s^2 ds \end{aligned}$$



$$= h f(x_0) + \frac{h^3}{24} f''(s) \cdot 2 \leq \frac{h^3}{24} |f''|$$

$$= h f(x_0) + \frac{h^3}{24} f''(s) h^3 \quad \text{LOCAL TRUNCATION ERROR}$$

COMPOSITE MID POINT RULE

$$\int_a^b f(x) dx = h [f_1 + \dots + f_N] + \frac{h^2}{24} \sum_{k=1}^N f''(s_k) h$$

$$= M_N(h) + \frac{h^3}{24} \int_a^b f(s) ds$$

$$= M_N(h) + \frac{h^2}{24} [f'(b) - f'(a)]$$

