Lecture 5

Piecwise Polynomial Approximation - Finite Element Basis Functions

Idea: Limit oscillations of high degree by restricting ourselves to low degree polynomials over subintervals.

Piecewise Linear Interpolation

DOF: 2N unknowns - (N-1) constraints C^0 continuity = N+1 DOF given by f_0, ..., f_N.

Piecewise Linear Lagrange Basis Functions - Hat Functions

\[ N_k^{(1)}(x) = \begin{cases} \frac{(x-x_{k-1})}{x_k-x_{k-1}} & x \in [x_{k-1}, x_k] \\ \frac{(x_{k+1}-x)}{x_{k+1}-x_k} & x \in [x_k, x_{k+1}] \\ 0 & \text{otherwise} \end{cases} \]

\[ N_0^{(1)}(x) = \begin{cases} \frac{(x-x_0)}{x_1-x_0} & x \in [x_0, x_1] \\ 0 & \text{otherwise} \end{cases} \]

\[ N_N^{(1)}(x) = \begin{cases} 0 & \text{otherwise} \end{cases} \]

\[ p_{N}^{(1)}(x) = \sum_{k=0}^{N} f_k N_k^{(1)}(x) \]

Error: \[ e(x) = f(x) - p_{N}^{(1)}(x) = (x-x_k)(x-x_{k+1}) \frac{f(x)}{2} \]

Now \( (x-x_k)(x-x_{k+1}) = (x_{k+1}-x_k)^2 \frac{1}{2} \] is a maximum at the midpoint \( x = x_k + \frac{x_{k+1} - x_k}{2} \)

\[ |e(x)| \leq \frac{1}{8} \| f \|_{\infty} \]
Representation on a Canonical Interval

It is often useful to perform calculations (e.g., integration) using these basis functions on a special (so-called canonical interval) \([-1, 1]\)

\[
N_1(s) = \frac{1}{2} (1 - s) \quad N_2(s) = \frac{1}{2} (1 + s)
\]

\[
N_0(s) = \frac{1}{2} (1 + s) \quad s_1 = -1, \quad s_2 = 1
\]

Typically, the interval \([x_k, x_{k+1}]\) is mapped onto the interval \([-1, 1]\) by the linear transformation

\[
x(s) = x_k N_1(s) + x_{k+1} N_2(s) = \frac{1}{2} (1 - s)x_k + \frac{1}{2} (1 + s)x_{k+1} = \left(\frac{x_k + x_{k+1}}{2}\right) + \left(\frac{x_{k+1} - x_k}{2}\right) s
\]

\[
\phi_i(s) = f_k N_i(s) + f_{k+1} N_2(s)
\]

Is called an isoparametric representation of the function \(f\).

Disequence Quadratic Interpolation

\[
p_2(x) = A_0 + B x + C x^2
\]

DoE: 3N unknowns \(A_k, B_k, C_k\), \(k = 1, \ldots, N\).

- Enforce continuity at interior nodes \(\rightarrow (N-1)\) constraints

\[
3N - (N-1) = N + 1 + N \rightarrow f_0, \ldots, f_N \text{ an extra sample of } f \text{ within each subinterval } f_{k-1} \rightarrow f_k
\]