LECTURE 34 TIME STEPPING SCHEMES FOR INITIAL VALUE ODES

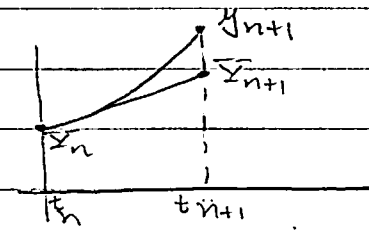
$$\dot{y} = f(t, y(t)) \qquad \text{INTEGRAL FORM } y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$$

$$y(0) = y_0 \in \mathbb{R}$$

THE FORWARD EULER METHOD

$$Y_{n+1} = Y_n + h f(t_n, Y_n)$$

$$Y_0 = y_0$$



GENERALIZATIONS OF EULER'S METHOD

FORWARD EULER

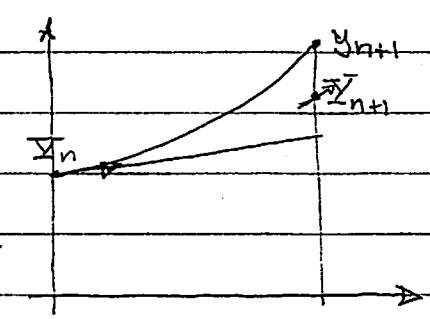
1-STEP	MULTI-STEP METHODS	
• IMPROVED EULER (RK2)	X LEAPFROG	X = EXPLICIT
• TRAPEZOIDAL SCHEME	X ADAMS-BASHFORTH	• = IMPLICIT.
• θ-METHOD	• ADAMS-MOULTON	
X MODIFIED EULER		
X RUNGE-KUTTA		

→ 1ST ORDER DIFFERENCE EQ → HIGHER ORDER DIFFERENCE EQ

SCHEMES BASED ON THE TRAPEZIUM RULE

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$$

$$= y_n + \frac{\Delta t}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right] + O(h^2)$$



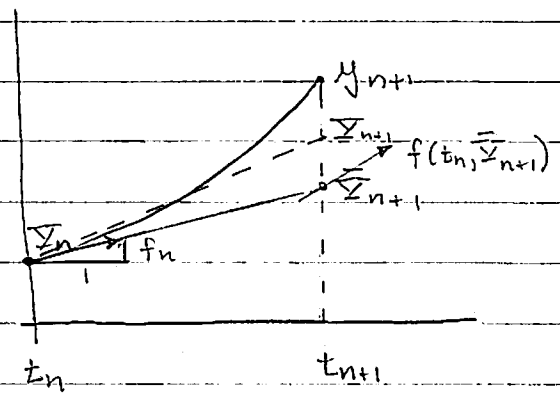
TRAPEZOIDAL SCHEME – CRANK-NICHOLSON SCHEME – IMPLICIT

$$\underbrace{Y_{n+1} = Y_n + \frac{\Delta t}{2}\left[f(t_n, Y_n) + f(t_{n+1}, Y_{n+1})\right]}_{\text{IMPLICIT}}$$

• SINCE THIS SCHEEME IS IMPLICIT YOU NEED TO SOLVE A NONLINEAR EQUATION (SYSTEM IF $y \in \mathbb{R}^N$) TO ADVANCE EACH TIME STEP.

THE IMPROVED EULER SCHEME – EXPLICIT

- PREDICTOR – CORRECTOR
- RUNGE – KUTTA 2 (RK2)

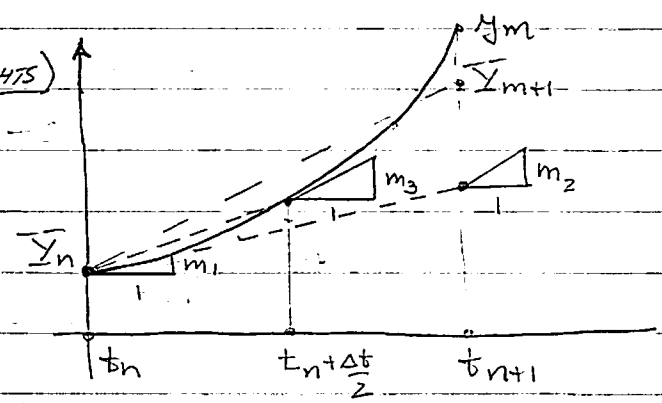


PREDICTOR: $\bar{\bar{y}}_{n+1} = \bar{y}_n + \Delta t f(t_n, \bar{y}_n)$ (EULER STEP)

CORRECTOR: $\bar{y}_{n+1} = \bar{y}_n + \frac{\Delta t}{2}\left[f(t_n, \bar{y}_n) + f(t_{n+1}, \bar{\bar{y}}_{n+1})\right]$

- SECOND ORDER ACCURATE
- EXPLICIT
- IF WE KEEP REPLACING $\bar{\bar{y}}_{n+1}$ BY $\bar{y}_{n+1}$ TILL CONVERGENCE WE OBTAIN THE TRAPEZOIDAL SCHEME
- FINAL STEP TAKES A WEIGHTED AVERAGE OF THE GRADIENTS OVER $[t_n, t_{n+1}]$

RUNGE – KUTTA SCHEMES (SIMPSON WEIGHTS)



R-K3: $\bar{y}_{n+1} = \bar{y}_n + \frac{\Delta t}{6}\left[m_1 + 4m_3 + m_2\right]$

$m_1 = f(t_n, \bar{y}_n)$

$m_2 = f(t_n + \Delta t, \bar{y}_n + \Delta t\, m_1)$

$m_3 = f\left(t_n + \frac{\Delta t}{2}, \bar{y}_n + \frac{\Delta t}{4}(m_1 + m_2)\right)$

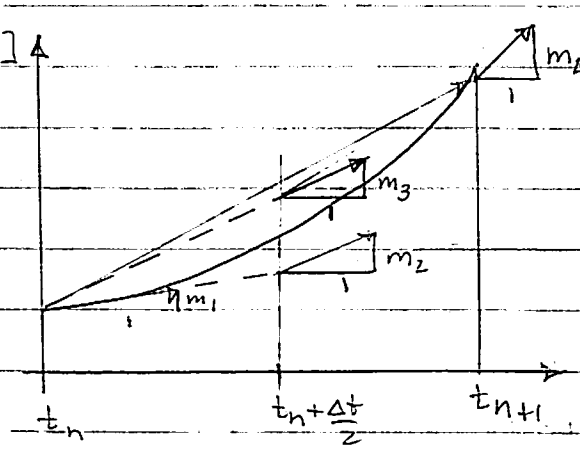
RK4: $\bar{y}_{n+1} = \bar{y}_n + \frac{\Delta t}{6}\left[m_1 + 2m_2 + 2m_3 + m_4\right]$

$m_1 = f(t_n, \bar{y}_n)$

$m_2 = f\left(t_n + \frac{\Delta t}{2}, \bar{y}_n + \frac{\Delta t}{2} m_1\right)$

$m_3 = f\left(t_n + \frac{\Delta t}{2}, \bar{y}_n + \frac{\Delta t}{2} m_2\right)$

$m_4 = f(t_n, \bar{y}_n + \Delta t\, m_3)$



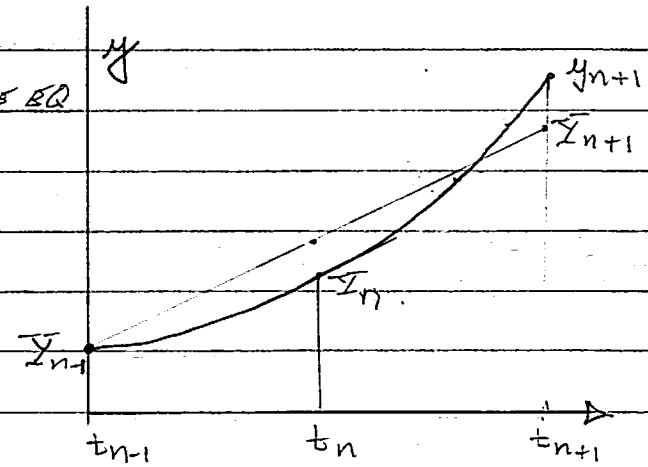
THE LEAPFROG SCHEME

- TWO-STEP → 2ND ORDER DIFFERENCE EQ
- EXPLICIT
- 2ND ORDER ACCURATE

$$y_{n+1} = y_n + \int_{t_{n-1}}^{t_{n+1}} f(s, y(s))\,ds$$

USING A MIDPOINT APPROXIMATION

$$Y_{n+1} = Y_{n-1} + 2\Delta t\, f(t_n, Y_n)$$



CONVERGENCE THEORY

THE CONVERGENCE THEORY FOR TIME-STEPPING SCHEMES IS BASED ON TWO CONCEPTS CONSISTENCY AND STABILITY

CONSISTENCY:

CONSISTENCY IMPLIES THE DIFFERENCE EQUATION (DCE) $\xrightarrow{\Delta t \to 0}$ ODE AS $\Delta t \to 0$

TRUNCATION ERROR IS THE LARGEST TERM THAT REMAINS WHEN YOU EXPAND EACH TERM IN THE DCE USING A TAYLOR SERIES AND PLUG IN THE EXACT SOLUTION TO THE ODE $\dot{y} = f(t, y)$.

EG EULERS METHOD: $\dfrac{Y_{n+1} - Y_n}{\Delta t} = f(t_n, Y_n)$

$$T_n = \frac{y_{n+1} - y_n}{\Delta t} - f(t_n, y_n) = \frac{\cancel{y_n} + \Delta t\, \dot{y}_n + \Delta t^2 \ddot{y}_n + \cdots - \cancel{y_n}}{\Delta t} - f(t_n, y_n) = O(\Delta t)$$

CONSISTENCY

A DIFFERENCE SCHEME IS CONSISTENT WITH A DIFFERENTIAL EQ IF THE TRUNCATION ERROR $T_n(\Delta t) \to 0$ AS $\Delta t \to 0$

CONVERGENCE:

A DIFFERENCE SCHEME CONVERGES AT A RATE $O(\Delta t^p)$ IF FOR Δt SUFFICIENTLY SMALL $|Y_n(\Delta t) - y(n\Delta t)| < C\Delta t^p$

STABILITY

STABILITY IMPLIES THAT ROUNDOFF ERRORS DO NOT GROW AS THE SOLUTION EVOLVES

ZERO-STABILITY: A DIFFERENCE SCHEME FOR WHICH PERTURBATIONS REMAIN BOUNDED IN THE LIMIT AS $\Delta t \to 0$ IS SAID TO BE 0-STABLE

TEST FOR 0-STABILITY AN N-STEP METHOD

$$a_N y_{n+N} + \cdots + a_0 y_n = \Delta t \left[b_0 f_n + b_1 f_{n+1} + \cdots + b_N f_{n+N} \right]$$

IS 0-STABLE IF THE ROOTS OF THE POLYNOMIAL

$$P_N(\theta) = a_N \theta^N + \cdots + a_1 \theta + a_0$$

ARE SUCH THAT $|\theta| \leq 1$ AND THOSE FOR WHICH $|\theta| = 1$ ARE SIMPLE

EG: EULER'S METHOD: $Y_{n+1} - Y_n = \Delta t \, f_n$

$P_1(\theta) = \theta - 1 = 0 \qquad \theta = 1 \Rightarrow$ 0-STABLE

EG: THE LEAPFROG SCHEME $Y_{n+2} - Y_n = \Delta t \, f_{n+1}$

$P_2(\theta) = \theta^2 - 1 = 0 \qquad \theta = \pm 1 \qquad$ 0-STABLE

EG: STRANGE SCHEME $Y_{n+2} - 2Y_{n+1} + Y_n = \Delta t (f_{n+2} - f_n)$

$P_2(\theta) = \theta^2 - 2\theta + 1 = (\theta - 1)^2 = 0 \qquad \theta = 1, 1$ 2 IS A DOUBLE ROOT

THEOREM: (DAHLQUIST) CONSISTENCY + 0-STABILITY $\Rightarrow$ CONVERGENCE

A 0-STABLE CONSISTENT SCHEME CONVERGES WITH THE ORDER OF ITS TRUNCATION ERROR