Lecture 34  Time Stepping Schemes for Initial Value ODEs

\[ \dot{y} = f(t, y(t)) \]  
**Internal Form** \[ \dot{y}(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s)) \, ds \]

\[ y(0) = y_0 \in \mathbb{R} \]

**The Forward Euler Method**

\[ \bar{y}_{n+1} = \bar{y}_n + h \, f(t_n, \bar{y}_n) \]

\[ y_0 = y_0 \]

**Generalizations of Euler's Method**

**Forward Euler**

\[ \begin{array}{ll}
\text{1-Step} & \text{Multi-Step Methods} \\
\hline
* Improved Euler (K1) & * Adams-Moulton
* Trapezoidal Scheme & * Adams-Bashforth
* θ-Methd &
* Modified Euler
* Runge-Kutta
\end{array} \]

\[ \rightarrow \text{1st Order Difference Eq.} \rightarrow \text{Higher Order Difference Eq.} \]

**Schemes Based on the Trapezium Rule**

\[ \bar{y}_{n+1} = \bar{y}_n + \frac{h}{2} \left[ f(t_n, \bar{y}_n) + f(t_{n+1}, \bar{y}_{n+1}) \right] + O(h^2) \]

**Trapezoidal Scheme - Crank-Nicholson Scheme - Implicit**

\[ \bar{y}_{n+1} = \bar{y}_n + \frac{h}{2} \left[ f(t_n, \bar{y}_n) + f(t_{n+1}, \bar{y}_{n+1}) \right] \]

\[ \text{Implicit} \]

* Since this scheme is implicit you need to solve a nonlinear equation (system if higher) to advance each time step.
THE IMPROVED EULER SCHEME - EXPLICIT

- PREDICTOR - CORRECTOR

- RUNGE-KUTTA 2 (RK2)

\[ \tilde{y}_{n+1} = \tilde{y}_n + \Delta t f(t_n, \tilde{y}_n) \]  
(BULKE Step)

\[ \tilde{y}_{n+1} = \tilde{y}_n + \Delta t \left[ \frac{f(t_n, \tilde{y}_n) + f(t_{n+1}, \tilde{y}_{n+1})}{2} \right] \]

- SECOND ORDER ACCURATE

- EXPLICIT

- IF WE KEEP REPLACING \( \tilde{y}_{n+1} \) BY \( \tilde{y}_n \) TILL CONVERGENCE WE OBTAIN THE
  TRAPEZOIDAL SCHEMES

- FINAL STEP TAKES A WEIGHTED AVERAGE OF THE GRADIENTS OVER \([t_n, t_{n+1}]\)

RUNGE-KUTTA SCHEMES (SIMPSON WEIGHTS)

\[ RK3: \quad \tilde{y}_{n+1} = \tilde{y}_n + \Delta t \left[ \frac{m_1 + 4m_3 + m_2}{6} \right] \]

\[ m_1 = f(t_n, \tilde{y}_n) \]

\[ m_2 = f(t_n + \Delta t, \tilde{y}_n + \Delta t m_1) \]

\[ m_3 = f(t_n + \frac{\Delta t}{2}, \tilde{y}_n + \frac{\Delta t}{2} (m_1 + m_2)) \]

\[ RK4: \quad \tilde{y}_{n+1} = \tilde{y}_n + \Delta t \left[ \frac{m_1 + 2m_2 + 2m_3 + m_4}{6} \right] \]

\[ m_1 = f(t_n, \tilde{y}_n) \]

\[ m_2 = f(t_n + \frac{\Delta t}{2}, \tilde{y}_n + \frac{\Delta t}{2} m_1) \]

\[ m_3 = f(t_n + \frac{\Delta t}{2}, \tilde{y}_n + \frac{\Delta t}{2} m_2) \]

\[ m_4 = f(t_n + \Delta t, \tilde{y}_n + \Delta t m_3) \]
**The Loadflow Scheme**

- **Two-Step → 2nd Order Difference Eq**
- **Explicit**
- **2nd Order Accurate**

\[
\begin{align*}
Y_{n+1} &= Y_n + \sum_{i=0}^{n-1} f(t_i, Y_i) \Delta s \\
Y'_{n+1} &= Y'_{n-1} + 2 \Delta t f(t_n, Y_n)
\end{align*}
\]

**Convergence Theory**

The convergence theory for time-stepping schemes is based on two concepts: consistency and stability.

**Consistency:**
Consistency implies the difference equation (DCE) \(\Delta t \to 0\) as \(\Delta \to 0\)

**Truncation Error** is the largest term that remains when you expand each term in the DCE using a Taylor series and plug in the exact solution to the ODE \(\dot{y} = f(t, y)\).

**Example Method:** \(\sum_{n+1} - \sum_n = f(t_n, Y_n)\)

\[
T_n = \frac{Y_{n+1} - Y_n - f(t_n, Y_n)}{\Delta t} = Y_n + \Delta t \frac{\Delta Y_n}{\Delta t} + \Delta t^2 \frac{\Delta^2 Y_n}{2!} + \cdots - f(t_n, Y_n) = O(\Delta t)
\]

**Consistency**
A difference scheme is consistent with a differential Eq if the truncation error \(T_n(\Delta t) \to 0\) as \(\Delta t \to 0\).

**Convergence:**
A difference scheme converges at a rate \(O(\Delta t^p)\) if for \(\Delta t\) sufficiently small, \(\mid \sum_{n}(\Delta t) - Y(n, \Delta t) \mid < C \Delta t^p\).
STABILITY

STABILITY implies that roundoff errors do not grow as the solution evolves.

ZERO-STABILITY: A difference scheme for which perturbations remain bounded in the limit as $\Delta t \to 0$ is said to be O-STEABLE.

TEST FOR O-STABILITY: A N-step method

$$0_n y_{n+1} + \ldots + a_0 y_n = \Delta t \left[ b_0 f_n + b_1 f_{n+1} + \ldots + b_n f_{n+i} \right]$$

Is O-STABLE if the roots of the polynomial

$$P_n(\theta) = a_n \theta^n + \ldots + a_1 \theta + a_0$$

are such that $|\theta| \leq 1$, and those for which $|\theta| = 1$ are simple.

EG: Euler’s method. $y_{n+1} - y_n = \Delta t f_n$

$$P_1(\theta) = \theta - 1 = 0 \quad \theta = 1 \implies \text{O-STABLE}$$

EG: The leapfrog scheme. $y_{n+2} - 2y_{n+1} + y_n = \Delta t f_{n+1}$

$$P_2(\theta) = \theta^2 - 1 = 0 \quad \theta = \pm 1 \quad \text{O-STABLE}$$

EG: Strang’s scheme. $y_{n+2} - 2y_{n+1} + y_n = \Delta t (f_{n+2} - f_n)$

$$P_2(\theta) = \theta^2 - 2\theta + 1 = (\theta - 1)^2 = 0 \quad \theta = 1, 1 \implies \text{A DOUBLE ROOT}$$

THEOREM: (DAHLQVIST) CONSISTENCY + O-STABILITY $\implies$ CONVERGENCE.

A O-STABLE CONSISTENT SCHEME CONVERGES WITH THE ORDER OF ITS TRUNCATION ERROR.