

## LECTURE 33

LAST TIME

$$U(r, \theta) = \int_0^{2\pi} g(\phi) \frac{\partial G}{\partial \rho} d\phi \quad r_1^2 = \rho^2 + r^2 - 2\rho r \cos(\phi - \theta)$$

$$r_2^2 = \rho^2 + R^2 - 2\rho R \cos(\phi - \theta)$$

$$G(\rho, \phi; r, \theta) = \frac{1}{2\pi} \ln\left(\frac{r_1}{r_2}\right) = \frac{1}{4\pi} \ln r_1^2 - \frac{1}{4\pi} \ln r_2^2 + \frac{1}{2\pi} \ln\left(\frac{a}{r}\right)$$

$$\frac{4\pi}{\partial \rho} \frac{\partial G}{\rho=a} = \frac{2\rho - 2r \cos \theta}{r_1^2} - \frac{2\rho - 2R \cos \theta}{r_2^2} \quad r_1 = \left(\frac{a}{r}\right)r, \quad R = \frac{a^2}{r}$$

$$= r_1^2 \left(\frac{a}{r}\right)^2 \frac{1}{r_1^2}$$

$$\frac{2\pi}{\partial \rho} \frac{\partial G}{\rho=a} = \frac{\frac{a^3}{r^2} - \frac{a^2}{r} \cos \theta}{\left(\frac{a}{r}\right)^2 r_1^2} - a + \frac{a^2}{r} \cos \theta$$

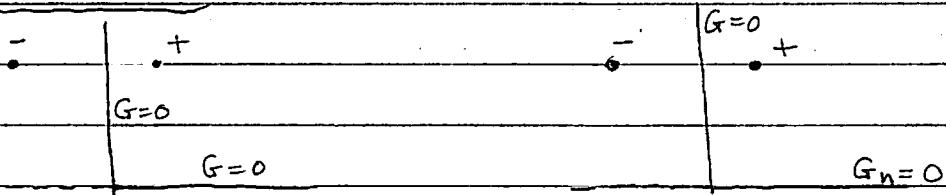
$$= \frac{a}{r^2} \frac{[a^2 - r^2]}{\left(\frac{a}{r}\right)^2 r_1^2}$$

$$= \frac{(a^2 - r^2)}{a r_1^2}$$

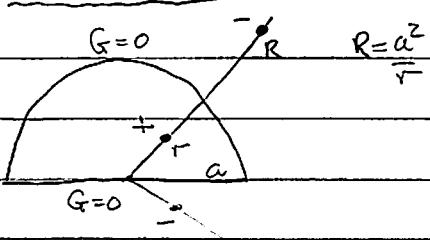
$$\therefore U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{g(\phi) (a^2 - r^2)}{a^2 + r^2 - 2ar \cos(\theta - \phi)} d\phi$$

OTHER GEOMETRIES:

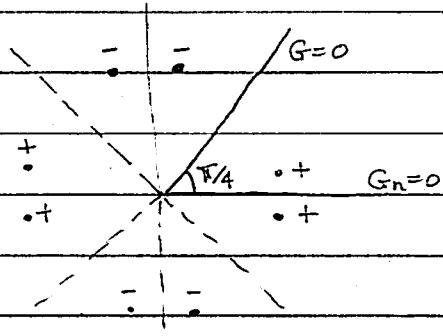
1. QUADRATIC PLANE



2. SEMICIRCLE



WEDGE



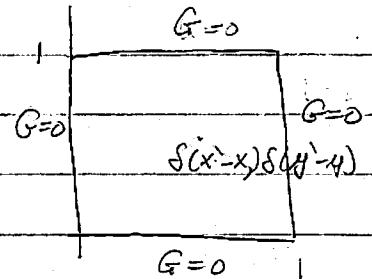
## EIGENFUNCTION EXPANSIONS

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### GREEN'S FUNCTION FOR A RECTANGULAR DOMAIN

$$\Delta_x G(x', y'; x, y) = \delta(x' - x) \delta(y' - y) \text{ ON } \Sigma.$$

$G = 0$  ON  $\partial\Sigma$



$$G(x', y'; x, y) = \sum_{m, n=1}^{\infty} G_{mn} \sin(m\pi x') \sin(n\pi y')$$

$$\delta(x' - x) \delta(y' - y) = \sum_{m, n=1}^{\infty} S_{mn} \sin(m\pi x) \sin(n\pi y)$$

$$S_{mn} = \frac{4}{1} \int_0^1 \int_0^1 \delta(x' - x) \delta(y' - y) \sin(m\pi x) \sin(n\pi y) dx' dy'$$

$$= 4 \sin(m\pi x) \sin(n\pi y)$$

$$\text{Now, } \Delta G = \sum_{m, n=1}^{\infty} = -[(m\pi)^2 + (n\pi)^2] G_{mn} \sin(m\pi x') \sin(n\pi y') = 4 \sum_{m, n=1}^{\infty} \sin(m\pi x) \sin(n\pi y)$$

$$\therefore G_{mn} = -\frac{4}{\pi^2(m^2 + n^2)}$$

$$\therefore G(x', y'; x, y) = -\frac{4}{\pi^2} \sum_{m, n=1}^{\infty} \frac{1}{m^2 + n^2} \sin(m\pi x) \sin(m\pi x') \sin(n\pi y) \sin(n\pi y')$$

## INITIAL VALUE PROBLEMS FOR ODE

CONSIDER

$$\begin{aligned}\dot{y} &= f(t, y(t)) \\ y(0) &= y_0\end{aligned}\quad \left. \begin{array}{l} (1) \\ \text{INITIAL CONDITION} \end{array} \right\} y \in \mathbb{R}^n$$

- IF  $f \in C^1$  THEN (1) HAS A UNIQUE SOLUTION
- THE BEHAVIOUR OF THE ERRORS IN THE NUMERICAL SOLUTION OF (1) IS RELATED TO THE BEHAVIOUR OF THE LINEARIZED PROBLEM.

LET  $\bar{y}(t)$  BE A NOMINAL SOLUTION AND  $\delta y$  A PERTURBATION THEN

$$\begin{aligned}y(t) &= \bar{y}(t) + \delta y(t) \\ \therefore \dot{y} &= (\dot{\bar{y}} + \dot{\delta y}) = f(t, \bar{y} + \delta y) = f(t, \bar{y}) + \frac{\partial f}{\partial y}(t, \bar{y}) \delta y + \dots\end{aligned}$$

$$\therefore \dot{\delta y} = \frac{\partial f}{\partial y}(t, \bar{y}) \delta y = J \delta y \quad \text{WHERE } J = \frac{\partial f}{\partial y} \text{ IS THE JACOBIAN.}$$

- ASSUME  $J$  IS CONSTANT IN TIME (FREEZE COEFFICIENTS) AND THAT  $J$  HAS  $N$  DISTINCT EIGENVALUES  $\lambda_j$  AND INDEPENDENT EIGENVECTORS  $v_j$  AND CHANGE VARIABLES  $\delta y = P z$  WHERE  $P = [v_1, v_2, \dots, v_N]$

THEN  $\dot{\delta y} = P \dot{z} = J P z$

OR  $\dot{z} = P^{-1} J P z = D z$  WHERE  $D = \begin{bmatrix} \lambda_1 \\ & \ddots \\ & & \lambda_N \end{bmatrix}$

$\therefore \dot{z}_j = \lambda_j z_j$

THUS WE DEFINING THE SCALAR MODEL PROBLEM

$$\dot{y} = \lambda y$$

$$y(0) = y_0$$

WHOSE EXACT SOLUTION IS  $y(t) = y_0 e^{\lambda t}$

NOTE: IF  $\operatorname{Re}(\lambda) < 0$  THE SOLUTION DECAYS EXPONENTIALLY  
IF  $\operatorname{Re}(\lambda) > 0$  THE SOLUTION GROWS EXPONENTIALLY

## SCHMIES TO SOLVE THE SCALAR INITIAL VALUE PROBLEM

$$\dot{y} = f(t, y(t)) \quad t \in \mathbb{R}$$

$$y(0) = y_0$$

### 1. TAYLOR SERIES

$$y(t_{n+1}) = y(t_n) + y'(t_n)\Delta t + \frac{y''(t_n)}{2!}\Delta t^2 + \dots + \frac{y^{(n)}(t_n)}{n!}\Delta t^n$$

$$\dot{y}_n = f(t_n, y_n)$$

$$\ddot{y} = f_t + f_y \dot{y} = f_t + f_y \dot{y}_n$$

$$\ddot{y}_n = [f_t + f_y \dot{y}_n]_{t=t_n}$$

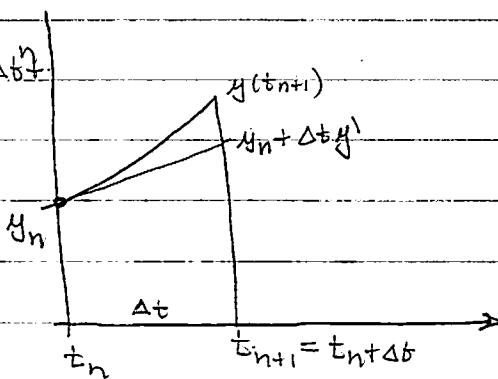
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$$\text{eg: } \ddot{y} = \lambda y \quad y(0) = y_0$$

$$\ddot{y} = \lambda \dot{y} = \lambda^2 y$$

$$y^{(n)} = \lambda^n y$$

$$\therefore y(t_{n+1}) = [1 + \frac{\Delta t \lambda}{1!} + \dots + \frac{(\Delta t \lambda)^n}{n!}] y(t_n) = e^{\lambda \Delta t} y(t_n) \quad \text{EXACT}$$



- BY TRUNCATING THE TAYLOR SERIES AT THE K-TH TERM WE OBTAIN AN APPROXIMATION OF  $O(\Delta t^k)$

- THE ACCURACY OF ALL NUMERICAL SCHEMES ARE MEASURED BY HOW MANY TERMS OF AGREEMENT THEY SHARE WITH THE TAYLOR SERIES

- MANY SCHEMES CAN BE INTERPRETED AS GIVING DIFFERENT APPROXIMATIONS TO  $e^{\Delta t \lambda}$  WHEN SOLVING THE MODEL PROBLEM

### 2. EULER'S METHOD

- TRUNCATE THE TAYLOR SERIES AFTER THE LINEAR TERM TO AVOID HAVING TO CALCULATE THE HIGHER DERIVATIVES:

$$\begin{aligned} y_{n+1} &= y_n + \Delta t \dot{y}_n + O(\Delta t^2) \\ &= y_n + \Delta t f(t_n, y_n) + O(\Delta t^2) \end{aligned}$$

FORWARD EULER

$$\Sigma_{n+1} = \Sigma_n + \Delta t f(t_n, \Sigma_n)$$

$$\Sigma_0 = y_0$$

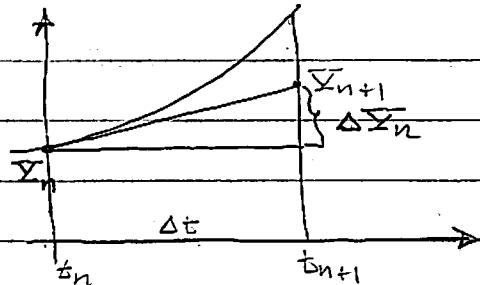
- THE FORWARD EULER METHOD IS EXPLICIT AS ALL THE INFORMATION AT THE N-TH STEP IS KNOWN IN ORDER TO PROCEED TO THE N+1 ST STEP.

### ALTERNATIVE DERIVATION - INTEGRAL FORM

$$\dot{y} = f(t, y(t))$$

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s)) ds$$

$$Y_{n+1} = Y_n + \Delta t f(t_n, Y_n)$$

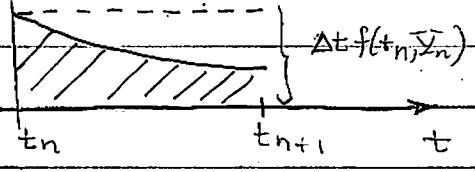


MANY STANDARD INTEGRATION RULES GIVE

RISE TO APPROXIMATION SCHEMES FOR ODES

VIA THIS INTEGRAL FORM

$$f(t_0, y(t))$$



LEFT HAND RULE