The Indirect Boundary Integral Method

Last time we considered the boundary integral equation

$$-u(x) + \int \frac{\partial v(x', x)}{\partial n'} ds(x') = \int q(x') \gamma(x', x) ds(x')$$

That is obtained when we let $x \to \partial \Omega$.

Considering a global coordinate system $Oxi$, in which the sending element has coordinates $(x_i, y_i)$ for its midpoint and orientation $\alpha_i$, the local coordinates $(x_j, y_j)$ of the midpoint $(x_i, y_i)$ of the jth receiving element are given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x_i - x_j \\ y_i - y_j \end{bmatrix}$$

Then defining $v_{ij} = V_i(x_i, y_i) = \frac{1}{2\pi} \ln \left( (x_i - x_j)^2 + y_i^2 \right)^{1/2} ds = \int_{S_i} \gamma(x', x_j) ds(x')$.

And

$$V_{Hij} = -\frac{\partial V_i}{\partial y}(x_i, y_i) = \frac{1}{2\pi} \left[ \tan^{-1} \left( \frac{y_i}{x_i - a} \right) - \tan^{-1} \left( \frac{y_i}{x_i + a} \right) \right]$$

Then

$$\sum_{i=1}^{N} g_{ij} q_i = -u_j + \sum_{i=1}^{N} V_{Hij} u_i = \sum_{i=1}^{N} h_{ij} u_i$$

Where $g_{ij} = v_{ij}$ and $h_{ij} = -\delta_{ij} + v_{Hij}$ so that $h_{ii} = \frac{-1}{2}$.

Thus

$$g q = h u$$

$$q = g^{-1} h u = g^{-1} H q$$

For the Dirichlet problem.
The Indirect Boundary Element Method - Single and Double Layer Potentials

Consider a closed curve $\Sigma_L$ that divides $\mathbb{R}^2$ into an interior region $\Sigma_I$ and an exterior region $\Sigma_E$. If $x \in \Sigma_I$,

$$u^i(x) = \int_{\Sigma_L} \frac{u^i(s')}{\|x-s\|} d\Sigma(s') - \int_{\Sigma_L} \frac{q^i(s') \nu(x,s')}{\|x-s\|} d\Sigma(s')$$

$$0 = \int_{\Sigma_L} \frac{u^e(s')}{\|x-s\|} d\Sigma(s') + \int_{\Sigma_L} \frac{q^e(s') \nu(x,s')}{\|x-s\|} d\Sigma(s')$$

Adding we obtain

$$\dot{u}(x) = \int_{\Sigma_L} \left( \delta u(x') - u^e(x') \right) \frac{\partial \nu(x',x)}{\partial n} d\Sigma'$$

$$- \int_{\Sigma_L} \left[ q_1(x') - q_2(x') \right] \nu(x',x) d\Sigma(x')$$

The flux jump to be non-zero $F(x) = [q] = [q_1 - q_2]$

Single layer potential: require $[u] = \{u^e - u^i\} = 0$ and allow the flux jump to be non-zero $F(x) = [q]$

Note: The flux can jump while the potential is continuous.

In this case the integral equation (4) reduces to

$$u(x) = \int_{\Sigma_L} E(x') \nu(x',x) d\Sigma(x')$$

Single layer potential.

This method has the advantage that only one integral is involved along the boundary. For example, for the Dirichlet problem $u = g$ on $\Sigma_L$ we have $q_1 = u(x) = \sum_{i=1}^N E_i \int_{\Sigma_L} \nu(x',x) d\Sigma(x') = GF \Rightarrow F = G^{-1}g$.

$F(x)$ is a fictitious quantity that represents the charge distribution along $\Sigma_L$ that generates the specified potential $g(x)$ on $\Sigma_L$. 
Double Layer Potential: Require \[ \phi_2 = \phi_1 \] \( \frac{q^i - q^j}{r} \) = 0 (Continuous)

Fluxes and allow the potential to jump across the boundary

\[ \text{i.e., let } D(x') = U^e(x') - U^i(x') = \left[ U \right] \]

In this case

\[ U^i(x) = - \int_{\partial\Sigma} D(x') \frac{\partial V(x',x)}{\partial n'} \, ds(x') \]

Note that \( D(x') \) can be interpreted as a dipole distribution

Along \( \partial\Sigma \) to achieve a potential \( U(x) = \varphi_0(x) \) along \( \partial\Sigma \)

\[ \lim_{\varepsilon \to 0} \frac{V(x,y+\varepsilon) - V(x,y-\varepsilon)}{2\varepsilon} = \frac{\partial V}{\partial n} = -\frac{\partial V}{\partial n} \]

Green's function for a point jump in \( u \):

\[ u_{xx} + u_{yy} = 0 \quad \frac{\hat{u}}{r} = \hat{u} \hat{y} = k^2 \hat{u} = 0 \quad k = \sqrt{k^2} \]

\[ [u] = 0 \quad [u] = \delta(x) \]

\[ \hat{u}(k,y) = \left\{ \begin{array}{ll} A(k)e^{-ky} & y < 0 \\ B(k)e^{ky} & y > 0 \end{array} \right. \]

\[ \hat{V}(k,y) = \left\{ \begin{array}{ll} -A(k)e^{-ky} & k > 0 \\ +B(k)e^{ky} & k < 0 \end{array} \right. \]

\[ I = \left[ \hat{u}(k,y) \right] = A(k) - B(k) \quad O = \left[ \hat{V}(k,0) \right] = -A(k) + B(k) \quad \Rightarrow B = -A \quad \& \quad A(k) = \frac{1}{2} \]

\[ \hat{u}(k,y) = \frac{1}{2} \text{sgn}(y) e^{-k|y|} \]

\[ u(x,y) = \frac{\text{sgn}(y)}{2\pi} \int_0^\infty e^{-k\sqrt{x^2+y^2}} \frac{1}{x} \cos(kx) \, dk = -\frac{1}{2\pi} \frac{y}{x^2+y^2} = \text{potential due to a point jump in } u. \]