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LECTURE 29 THE FREE SPACE GREEN'S FUNCTION

IDEA: SINCE CONSTRUCTING GREEN'S FUNCTIONS FOR ARBITRARY DOMAINS IS NOT POSSIBLE, WE LOOK FOR A SOLUTION TO THE PDE WITH JUST THE δ -FUNCTION SOURCE WITHOUT TAKING THE BOUNDARIES INTO ACCOUNT. THIS SOLUTION IS KNOWN AS THE FREE SPACE GREEN'S FUNCTION.

1. THE FREE SPACE GREEN'S FUNCTION FOR THE LAPLACIAN IN 2D.

$$\Delta V = \delta(x) \quad x \in \mathbb{R}^2$$

METHOD 1: EXPLOITING RADIAL SYMMETRY

IN POLAR COORDINATES THE EQUATION FOR V BECOMES

$$V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} = \delta(r)$$

SINCE V SHOULD BE INDEPENDENT OF θ , THE HOMOGENEOUS EQ ^{2πY} FOR V .

REDUCES TO $r^2 V_{rr} + r V_r = 0$,

WHICH IS A CAUCHY-EULER EQ FOR WHICH $V = r^\gamma$ IMPLIES $\gamma(\gamma-1) + \gamma = \gamma^2 = 0$

$$\text{THUS } V(r) = A \ln r + B$$

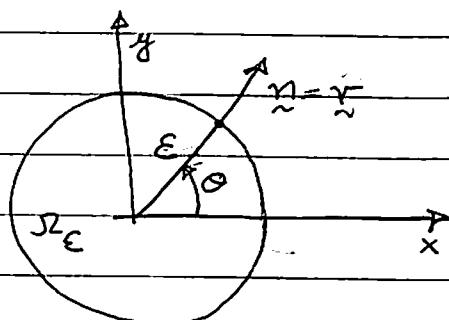
WE CHOOSE $B=0$ AND DETERMINE A TO MATCH THE STRENGTH OF THE SOURCE.

$$\text{INTEGRATE } \Delta V = \delta(r)/2\pi r \text{ OVER A}$$

CIRCULAR DISK Σ_ϵ OF RADIUS ϵ :

$$\int_{\Sigma_\epsilon} \Delta V dA = \int_{\partial\Sigma_\epsilon} \frac{\partial V}{\partial r} dr = \int_0^{2\pi} \int_0^\epsilon \frac{\delta(r)}{2\pi r} r dr d\theta$$

$$\therefore \int_0^{2\pi} \frac{A}{2\pi} \epsilon \theta d\theta = A 2\pi = 1$$



$$\therefore V(r) = \frac{1}{2\pi} \ln r$$

IN CARTESIAN COORDINATES THE SOLUTION TO

$$\Delta V(x, x) = \delta(x - x)$$

$$15 \quad V(x, x) = \frac{1}{2\pi} \ln \sqrt{(x-x)^2 + (y-y)^2}$$

METHOD 2: USING THE ONE DIMENSIONAL FOURIER TRANSFORM AND STITCHING

$$\Delta V = V_{xx} + V_{yy} = \delta(x)\delta(y) \quad (1)$$

LET $\hat{V}(k, y) = \int_x^{\infty} V(x, y) dx = \int_{-\infty}^{\infty} e^{ikx} V(x, y) dx$

BE THE x FOURIER TRANSFORM OF V . RECALL THAT

$$\mathcal{F}\{V_x\} = V e^{ikx} - ik \int_{-\infty}^{\infty} e^{ikx} V dx = -ik \hat{V}(k, y) \text{ ASSUMING } V \rightarrow 0 \text{ AS } x \rightarrow \infty.$$

TAKING THE FOURIER TRANSFORM OF BOTH SIDES OF (1) WE OBTAIN THE ODE

$$\hat{V}_{yy} - k^2 \hat{V} = \delta(y) \quad (2)$$

THE SOLUTION TO THE HOMOGENEOUS EQUATION IS OF THE FORM

$$\hat{V}(k, y) = A e^{-ky} + B e^{ky} \text{ WHERE } k = \sqrt{k^2}$$

IDENTIFYING THE SOLUTIONS THAT ARE BOUNDED AS $y \rightarrow \pm \infty$ WE OBTAIN

$$\hat{V}(k, y) = \begin{cases} A_+ e^{-ky} & y > 0 \\ B_- e^{ky} & y < 0 \end{cases}$$

IMPOSE CONTINUITY AT $y=0$: $\hat{V}(k, 0+) = A_+ = \hat{V}(k, 0-) = B_- \Rightarrow B_- = A_+$

THE APPROPRIATE JUMP CONDITION IS OBTAINED BY INTEGRATING (2) OVER $y \in (-\varepsilon, \varepsilon)$

$$\int_{-\varepsilon}^{\varepsilon} \hat{V}_{yy} - k^2 \hat{V} dy = \hat{V}(k, y) \Big|_{-\varepsilon}^{\varepsilon} = \hat{V}_y(k, 0+) - \hat{V}_y(k, 0-) = 1.$$

$$\therefore -A_+ k - A_+ k = 1 \Rightarrow A_+ = -1/2k$$

$$\text{THUS } \hat{V}(k, y) = -\frac{e^{-ky}}{2|k|}$$

INVERTING THE FOURIER TRANSFORM WE OBTAIN

$$V(x, y) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-|k|y}}{|k|} e^{-ikx} dk$$

$$= -\frac{1}{2\pi} \int_0^{\infty} \frac{e^{-|k|y}}{|k|} \cos kx dk$$

$$\text{NOW } V_y(x, y) = \frac{\operatorname{sgn}(y)}{2\pi} \int_0^{\infty} e^{-|k|y} \cos kx dk \text{ WHICH IS JUST A LAPLACE TRANSFORM}$$

$$= \frac{\operatorname{sgn}(y)}{2\pi} \operatorname{Re} \left\{ \int_0^{\infty} e^{k(-|y|+ix)} dk \right\}$$

$$= \frac{\operatorname{sgn}(y)}{2\pi} \operatorname{Re} \left\{ e^{k(-|y|+ix)} \Big|_0^{\infty} \right\}$$

$$= -\frac{\operatorname{sgn}(y)}{2\pi} \operatorname{Re} \left\{ \frac{-|y|-ix}{x^2+y^2} \right\} = \frac{y}{2\pi(x^2+y^2)}$$

$$\therefore V(x, y) = \frac{1}{2\pi} \ln [x^2+y^2]^{1/2}$$

2. THE FREE SPACE GREEN'S FUNCTION FOR THE LAPLACIAN IN 3D.

$$\Delta V = \delta(x) \quad x \in \mathbb{R}^3 \quad (1)$$

METHOD 1

IN SPHERICAL COORDINATES THE EQUATION FOR V BECOMES

$$\Delta V = V_{rr} + \frac{2}{r} V_r + \frac{1}{r^2} (V_{\theta\theta} + \cot\theta G_\theta + \cosec^2\theta G_{\phi\phi}) = \frac{\delta(r)}{4\pi r} \quad (2)$$

SINCE V SHOULD BE INDEPENDENT OF θ AND ϕ THE HOMOGENEOUS EQUATION FOR V REDUCES TO THE CAUCHY-GUILLEMIN EQ

$$r^2 V_{rr} + 2r V_r = 0$$

$$V = r^\gamma \Rightarrow \gamma(\gamma-1) + 2\gamma = \gamma^2 + \gamma = \gamma(\gamma+1) = 0 \quad \gamma = 0, -1$$

$$\therefore V = \frac{A}{r} + B \Rightarrow V_r = -A/r^2$$

CHOOSING $B=0$ WE DETERMINE A BY INTEGRATING (2) OVER A SPHERE S_E OF RADIUS ε :

$$1 = \int_{S_E} \Delta V d\sigma = \int_{S_E} \frac{\partial V}{\partial r} ds = \int_0^{2\pi} \int_0^\pi -A \cdot \varepsilon^2 \sin\phi d\theta d\phi = -4\pi A$$

$$\therefore V(r) = -\frac{1}{4\pi r}$$

METHOD 2: USING THE TWO DIMENSIONAL FOURIER TRANSFORM

$$V_{xx} + V_{yy} + V_{zz} = \delta(x) \delta(y) \delta(z)$$

$$\text{LET } \hat{V}(m, n, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(x, y, z) e^{imx+iny} dx dy.$$

$$\text{THEN } \hat{V}_{zz} - k^2 \hat{V} = \delta(z) \text{ WHERE } k = \sqrt{m^2+n^2}$$

$$\text{AS BEFORE } \hat{V}(m, n, z) = -\frac{e^{-k|z|}}{2k}$$

$$\therefore V(x, y, z) = -\frac{1}{2(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(mx+ny)} \frac{e^{-k|z|}}{k} dm dn$$

$$= -\frac{1}{8\pi^2} \int_0^{\infty} \int_{-\pi}^{\pi} e^{-ikr \cos(\theta-\theta_r)} \frac{e^{-k|z|}}{k} dk d\theta \quad mx+ny = kr \cos(\theta-\theta_r)$$

$$= -\frac{1}{8\pi^2} \int_0^{\infty} \int_{-\pi/2}^{\pi/2} e^{+ikr \sin(\bar{\theta}-\theta_r)} \frac{e^{-k|z|}}{k} dk d\bar{\theta} \quad \theta = \bar{\theta} + \pi/2 \quad \bar{\theta} \in [-\frac{3\pi}{2}, \frac{\pi}{2}]$$

$$= -\frac{1}{4\pi} \int_0^{\infty} \left\{ \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{ikr \sin t} dt \right\} \frac{e^{-k|z|}}{k} dk \quad t = \bar{\theta} - \theta_r$$

$$= -\frac{1}{4\pi} \int_0^{\infty} J_0(kr) e^{-k|z|} dk$$

$$= -\frac{1}{4\pi \sqrt{x^2+y^2+z^2}}$$

$$\text{SINCE } \int_0^{\infty} e^{-k|z|} J_0(kr) dk = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

