

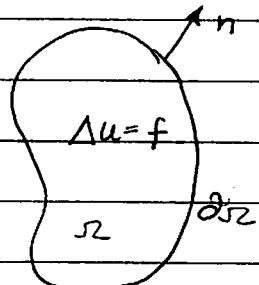
OUTLINE:

1. INTEGRAL IDENTITIES AND INTEGRAL REPRESENTATION OF SOLUTIONS
2. δ FUNCTION IN MULTIPLE DIMENSIONS AND CURVILINEAR COORDINATES
3. FREE SPACE GREEN'S FUNCTIONS - JUST THE SINGULAR SOURCE & NO BOUNDARIES
 - RADIAL SOLUTION
 - USING THE FT
4. INTEGRAL REPRESENTATION OF THE SOLUTION USING THE FREE SPACE GREEN'S FUNCTION - BOUNDARY INTEGRAL METHODS
5. THE METHOD OF IMAGES
 - USING THE FREE SPACE GREEN'S FUNCTION TO CONSTRUCT THE ACTUAL GREEN'S FUNCTION FOR PROBLEMS WITH REGULAR BOUNDARIES
6. EIGENFUNCTION EXPANSIONS & FOURIER TRANSFORMS

GREEN'S SECOND IDENTITY

- RECALL THE DIVERGENCE THEOREM

$$\int_{\Omega} \nabla \cdot \underline{F} d\underline{v} = \int_{\partial\Omega} \underline{n} \cdot \underline{F} ds$$



$$\nabla \cdot (\underline{u}_1 \nabla \underline{u}_2) = \nabla \underline{u}_1 \cdot \nabla \underline{u}_2 + \underline{u}_1 \nabla^2 \underline{u}_2$$

$$\begin{aligned} \langle \underline{u}_1, \Delta \underline{u}_2 \rangle &= \int_{\Omega} \underline{u}_1 \nabla^2 \underline{u}_2 d\underline{v} = \int_{\Omega} \nabla \cdot (\underline{u}_1 \nabla \underline{u}_2) - \nabla \underline{u}_1 \cdot \nabla \underline{u}_2 d\underline{v} \\ &= \int_{\partial\Omega} \underline{u}_1 \frac{\partial \underline{u}_2}{\partial \underline{n}} ds - \int_{\Omega} \nabla \underline{u}_1 \cdot \nabla \underline{u}_2 d\underline{v} \end{aligned}$$

$$\text{SIMILARLY } \langle \underline{u}_2, \Delta \underline{u}_1 \rangle = \int_{\partial\Omega} \underline{u}_2 \frac{\partial \underline{u}_1}{\partial \underline{n}} ds - \int_{\Omega} \nabla \underline{u}_2 \cdot \nabla \underline{u}_1 d\underline{v}$$

$$\therefore \boxed{\langle \underline{u}_1, \Delta \underline{u}_2 \rangle - \langle \underline{u}_2, \Delta \underline{u}_1 \rangle = \int_{\partial\Omega} [\underline{u}_1 \frac{\partial \underline{u}_2}{\partial \underline{n}} - \underline{u}_2 \frac{\partial \underline{u}_1}{\partial \underline{n}}] ds}$$

THIS IS REMINISCENT OF LAGRANGE'S IDENTITY IN WHICH

$$\langle \underline{u}_1, L \underline{u}_2 \rangle - \langle \underline{u}_2, L^* \underline{u}_1 \rangle = \text{BOUNDARY TERMS}$$

THE GREEN'S FUNCTION FOR THE DIRICHLET PROBLEM AND
AN INTEGRAL REPRESENTATION OF THE SOLUTION.

CONSIDER THE DIRICHLET PROBLEM

$$\Delta u = f \text{ ON } \Omega$$

$$u = g \text{ ON } \partial\Omega$$

LET $u_1 = u$ AND $u_2 = G$ IN GREEN'S SECOND IDENTITY

THEN

$$\int_{\Omega} u \Delta G \, dv = \int_{\Omega} G f \, dv + \int_{\partial\Omega} g \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \, ds$$

UNKNOWN

$$\Delta u = f$$

$$u = g$$

$$\partial\Omega$$

NOW LET $\Delta_s G(s, x) = \delta(s-x)$ ON Ω $x, s \in \mathbb{R}^n$

AND $G(s, x) = 0$ FOR $s \in \partial\Omega$

THEN

$$u(x) = \int_{\Omega} G(s, x) f(s) \, dv(s) + \int_{\partial\Omega} g(s) \frac{\partial G(s, x)}{\partial n} \, ds(s)$$

THUS THE GREEN'S FUNCTION YIELDS AN INTEGRAL REPRESENTATION OF THE SOLUTION

THE GREEN'S FUNCTION FOR A MIXED BOUNDARY VALUE PROBLEM

$$\Delta u = f \text{ ON } \Omega \quad u = g \text{ ON } \partial\Omega \quad \frac{\partial u}{\partial n} = h \text{ ON } \partial\Omega_h$$

$$(u, \Delta G) - (G, f) = \int_{\partial\Omega_g} u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \, ds$$

$$+ \int_{\partial\Omega_h} u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \, ds$$

IF WE NOW CHOOSE $G(s, x)$ TO BE THE SOLUTION
OF THE BOUNDARY VALUE PROBLEM

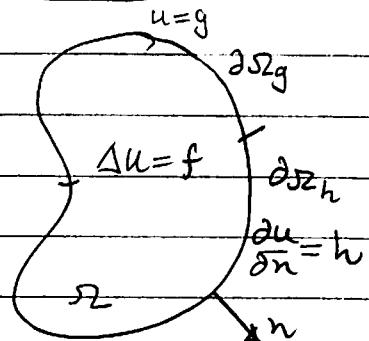
$$\Delta_s G(s, x) = \delta(s-x) \quad s \in \Omega$$

$$G(s, x) = 0 \text{ FOR } s \in \partial\Omega_g \text{ AND}$$

$$\frac{\partial G}{\partial n}(s, x) = 0 \text{ FOR } s \in \partial\Omega_h$$

THEN

$$u(x) = \int_{\Omega} G(s, x) f(s) \, dv(s) + \int_{\partial\Omega_g} g(s) \frac{\partial G(s, x)}{\partial n} \, ds(s) - \int_{\partial\Omega_h} G(s, x) h(s) \, ds(s)$$



2. MULTI DIMENSIONAL DELTA FUNCTIONS IN CURVILINEAR COORDINATES

CARTESIANS: $\int \delta(x-x', y-y') f(x', y') dx' dy' = \int \delta(x-x) \delta(y-y) f(x', y') dx' dy' = f(x, y)$

POLAR COORDINATES:

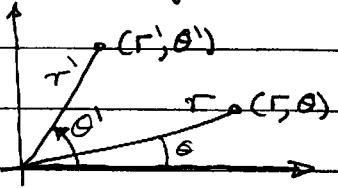
$$f(x, y) = \int \delta(x-x', y-y') f(x', y') dx' dy'$$

$$= \int \delta(r-r', \theta-\theta') \tilde{f}(r', \theta') r' dr' d\theta' \quad \text{LET } \tilde{f}(r', \theta') = f(r' \cos \theta', r' \sin \theta')$$

$$= \int \delta(r-r') \delta(\theta-\theta') \tilde{f}(r', \theta') dr' d\theta'$$

$$= \tilde{f}(r, \theta) \quad \text{THUS}$$

$$\delta(r-r', \theta-\theta') = \frac{\delta(r-r') \delta(\theta-\theta')}{r'}$$



SINGULAR POINT:

WHAT HAPPENS IF $r=0$? THEN $\tilde{f}(0, \theta)$ CANNOT DEPEND ON θ IF f IS SINGLE VALUED.

$$f(0, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) f(x, y) dx dy$$

$$= \int_{-\pi}^{\pi} \int_0^{\infty} \delta(r', \theta') f(r' \cos \theta', r' \sin \theta') r' dr' d\theta'$$

$$= \int_{-\pi}^{\pi} \int_0^{\infty} \frac{\delta(r')}{2\pi r'} f(r', \cos \theta', \sin \theta') r' dr' d\theta' = \frac{2\pi}{2\pi} f(0, \theta)$$

$$\therefore \delta(r', \theta') = \frac{\delta(r')}{2\pi r'}$$

SPHERICAL COORDINATES

$$f(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-x', y-y', z-z') f(x', y', z') dx' dy' dz'$$

$$= \int_{-\pi}^{\pi} \int_0^{\pi} \int_0^{\infty} \delta(r'-r, \theta'-\theta, \phi'-\phi) f(r' \sin \theta' \cos \phi', r' \sin \theta' \sin \phi', r' \cos \theta') r'^2 \sin \phi' dr' d\phi' d\theta'$$

$$= \int \delta(r-r') \delta(\theta-\theta') \delta(\phi-\phi') \tilde{f}(r', \theta', \phi') r'^2 \sin \phi' dr' d\phi' d\theta' = \tilde{f}(r, \theta, \phi)$$

FOR THE SINGULAR POINT $r=0$ $\delta(r', \theta', \phi') = \frac{\delta(r')}{4\pi r'^2}$

