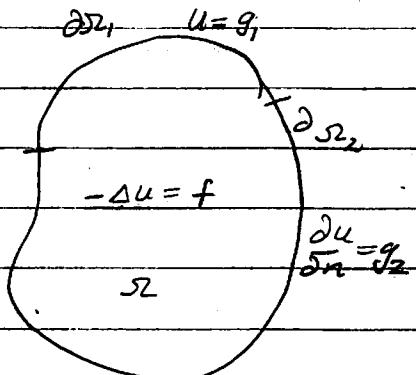


## LECTURE 25 THE LAPLACIAN IN 2D - POISSON, EIGENVALUES AND HELMHOLTZ PROBLEMS

### POISSON PROBLEM

$$\begin{aligned} -\Delta u = f & \quad \text{ON } \partial\Omega \\ \text{ESSENTIAL BC: } u = g_1 & \quad \text{ON } \partial\Omega_1 \\ \text{NATURAL BC: } \frac{\partial u}{\partial n} = g_2 & \quad \text{ON } \partial\Omega_2 \end{aligned} \quad \left. \right\} (5)$$



$$\text{WR: } \int_{\Omega} v \{ \nabla^2 u + f \} d\omega = 0 \quad \forall v \in C^0(\bar{\Omega})$$

$$\text{Now } \nabla \cdot (\nabla v) = \nabla^2 v + \nabla v \cdot \nabla v \Rightarrow \nabla^2 v = \nabla \cdot (\nabla v) - \nabla v \cdot \nabla v$$

$$\therefore \int_{\Omega} v \nabla^2 u d\omega = \int_{\Omega} \nabla \cdot (\nabla v) u d\omega - \int_{\Omega} \nabla v \cdot \nabla u d\omega = \int_{\Omega} v \frac{\partial u}{\partial n} ds - \int_{\partial\Omega} \nabla v \cdot \nabla u d\omega$$

|| ESSENTIAL BC      || NATURAL BC

$$\therefore \int_{\partial\Omega_1} v \frac{\partial u}{\partial n} ds + \int_{\partial\Omega_2} v \frac{\partial u}{\partial n} ds - \int_{\Omega} \nabla v \cdot \nabla u d\omega + \int_{\Omega} f v d\omega = 0$$

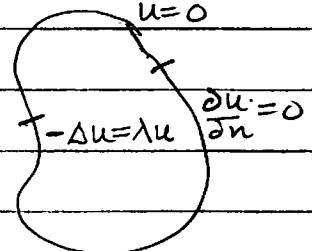
GIVEN  $f, g_1, g_2$  FIND  $u \in H_0^1 = \{u: u = g_1 \text{ ON } \partial\Omega_1 \text{ AND } \int_{\Omega} |\nabla u|^2 d\omega < \infty\}$  (W)

$$\text{SUCH THAT } a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v d\omega = \int_{\partial\Omega_2} v g_2 ds + \int_{\Omega} f v d\omega \quad \forall v \in H_0^1,$$

### EIGENVALUE PROBLEM:

$$\Delta u + \lambda u = 0 \quad \text{ON } \partial\Omega$$

$u = 0 \text{ ON } \partial\Omega_1 \quad \frac{\partial u}{\partial n} = 0 \text{ ON } \partial\Omega_2$



$$\int_{\Omega} v \{ \nabla^2 u + \lambda u \} d\omega = 0$$

|| ESSENTIAL BC      || NATURAL

$$\int_{\partial\Omega_1} v \frac{\partial u}{\partial n} ds + \int_{\partial\Omega_2} v \frac{\partial u}{\partial n} ds - \int_{\Omega} \nabla u \cdot \nabla v d\omega + \lambda \int_{\Omega} u v d\omega$$

FIND  $u \in H_0^1 = \{u: u = 0 \text{ ON } \partial\Omega_1 \text{ AND } \int_{\Omega} |\nabla u|^2 d\omega < \infty\}$  AND  $\lambda$  SUCH THAT

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v d\omega = \lambda \int_{\Omega} u v d\omega, \quad \forall v \in H_0^1,$$

### GALERKIN APPROXIMATION TO THE POISSON PROBLEM

LET  $U^h(x, y) = \sum_{n=1}^N u_n \psi_n(x, y) + \sum_{s=1}^S g_{1s} \psi_s(x, y)$   $(x_n, u_n) \in \Omega \cup \partial\Omega_2$   
 $(x_s, g_{1s}) \in \partial\Omega_1$

FOR PIECEWISE LAGRANGE POLYNOMIALS  $\psi_p(x_q, y_q) = \delta_{pq}$

$$V^h(x, y) = \sum_{m=1}^N v_m \psi_m \quad (x_m, v_m) \in \Omega \cup \partial\Omega_2$$

$$\int_{\Omega} \nabla \left( \sum_{n=1}^N u_n \psi_n + \sum_{s=1}^S g_{1s} \psi_s \right) \cdot \nabla \left( \sum_{m=1}^N v_m \psi_m \right) d\Omega$$

$$= \int_{\partial\Omega_2} \left( \sum_{m=1}^N v_m \psi_m \right) g_2 ds + \int_{\Omega} \sum_{m=1}^N v_m \psi_m f d\Omega$$

$$\therefore \sum_{m=1}^N v_m \left\{ \sum_{n=1}^N u_n \int_{\Omega} \nabla \psi_n \cdot \nabla \psi_m d\Omega + \sum_{s=1}^S g_{1s} \int_{\Omega} \nabla \psi_s \cdot \nabla \psi_m d\Omega \right.$$

$$\left. - \int_{\partial\Omega_2} \psi_m g_2 ds - \int_{\Omega} \psi_m f d\Omega \right\} = 0$$

OR

$$\sum_{n=1}^N A_{mn} u_n = b_m$$

WHERE

$$A_{mn} = \int_{\Omega} \nabla \psi_m \cdot \nabla \psi_n d\Omega = A_{nm} \quad \text{SYMMETRIC STIFFNESS MATRIX.}$$

AND

$$b_m = \int_{\partial\Omega_2} \psi_m g_2 ds + \int_{\Omega} \psi_m f d\Omega - \sum_{s=1}^S g_{1s} \int_{\Omega} \nabla \psi_s \cdot \nabla \psi_m d\Omega$$

### EIGENVALUE PROBLEM:

LET  $U^h(x, y) = \sum_{n=1}^N u_n \psi_n(x, y)$  AND  $V^h = \sum_{n=1}^N v_n \psi_n(x, y)$

WHERE  $\psi_n(x, y) = 0$  ON  $\partial\Omega_1$  THEN

$$\sum_{n=1}^N u_n \int_{\Omega} \nabla \psi_n \cdot \nabla \psi_m d\Omega = \lambda \sum_{n=1}^N u_n \int_{\Omega} \psi_n \psi_m d\Omega \quad m=1, \dots, N$$

OR  $\sum_{n=1}^N A_{mn} u_n = \lambda \sum_{n=1}^N M_{mn} u_n \quad A u = \lambda M u$

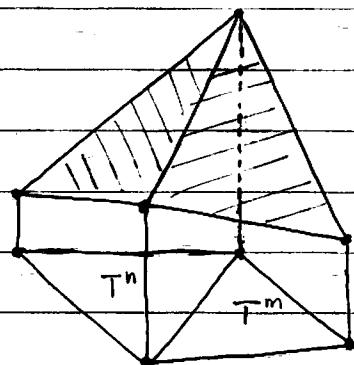
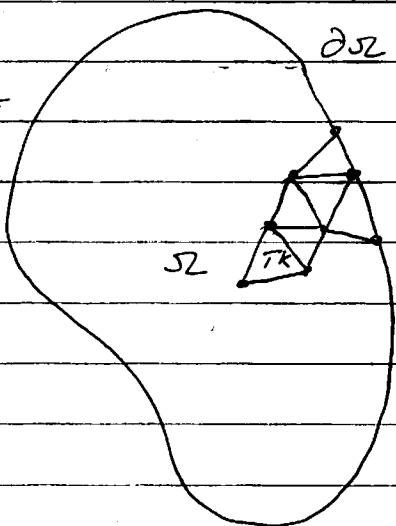
WHERE

$$A_{mn} = \int_{\Omega} \nabla \psi_n \cdot \nabla \psi_m d\Omega = A_{nm} \quad \text{IS THE STIFFNESS MATRIX}$$

AND  $M_{mn} = \int_{\Omega} \psi_n \psi_m d\Omega = M_{mm} \quad \text{IS THE MASS MATRIX}$

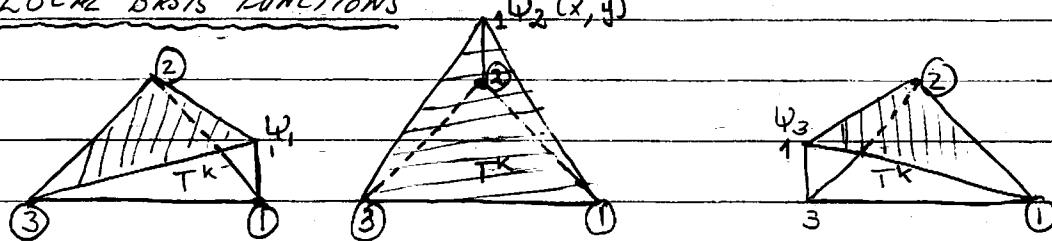
## DISCRETIZATION USING PIECEWISE LINEAR FUNCTIONS DEFINED ON TRIANGLES

- TESSELATE THE DOMAIN  $S_L$  INTO  $N$  TRIANGULAR ELEMENTS  $T^K$   $K=1, \dots, N$  SUCH THAT
$$S_L \approx \bigcup_{K=1}^N T^K$$
- CONSIDER THE SPACE OF PIECEWISE LINEAR FUNCTIONS THAT ARE CONTINUOUS ALONG THE BOUNDARY SEGMENTS



- CONSTRUCT BASIS FUNCTIONS  $\psi_k(x, y)$  SO THAT  $\psi_k(x_j, y_j) = \delta_{kj}$

LOCAL BASIS FUNCTIONS



$$\text{LET } \psi_a^k(x, y) = c_a^k + c_{x,a}^k x + c_{y,a}^k y$$

$$\begin{array}{c|c}
V & G \\
\hline
\begin{matrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{matrix} & \begin{matrix} c_1^k & c_2^k & c_3^k \\ c_{x,1}^k & c_{x,2}^k & c_{x,3}^k \\ c_{y,1}^k & c_{y,2}^k & c_{y,3}^k \end{matrix} \\
\hline
& = I
\end{array}$$

(VAN DE MONDE MATRIX)

$$G = V^{-1}$$

