

LECTURE 18:

EG 2:  $Lu = u'' + u = f(x), u(0) = 0 = u(\pi)$  (1)

- 1) DETERMINE THE SOLVABILITY CONDITION FOR  $f$
- 2) DETERMINE THE MODIFIED GREEN'S FUNCTION FOR (1)

NOTE THAT  $\tilde{u}(x) = \sin x$  IS A NONTRIVIAL SOLUTION TO  $Lu = 0 + BC$ .

1) THE SOLVABILITY CONDITION ON  $f$  IS

$$\int_0^\pi f(x) \sin x dx = 0$$

$$2) (v, Lu) = \int_0^\pi v \{u'' + u\} dx = [vu' - v'u]_0^\pi + \int_0^\pi u Lv dx$$

$$= v(\pi)u'(\pi) - v'(0)u(0) - v'(\pi)u(\pi) + v(0)u'(0) + \int_0^\pi u Lv dx$$

THE MODIFIED GREEN'S FCN SHOULD SATISFY

$$L_s G(s, x) = G_{ss} + G = \delta(s-x) - \sin s \sin x / \int_0^\pi \sin^2 s ds = \delta(s-x) - \frac{2 \sin s \sin x}{\pi}$$

$$G(0, x) = 0 \text{ AND } G(\pi, x) = 0 \quad C = -\frac{2 \sin x}{\pi}$$

PARTICULAR SOLN TO  $v_{ss} + v = C \sin s$  IS  $v(s) = -\frac{C \cos s}{2} = \frac{2 \cos s \sin x}{\pi}$

LET  $\tilde{G}(s, x) = \frac{2 \cos s \sin x}{\pi} + \begin{cases} A_- \sin s & 0 < s < x \\ A_+ \sin s + B_+ \cos s & x < s < \pi \end{cases}$

$\tilde{G}(0, x) = 0$   
 $\tilde{G}(\pi, x) = -\cos \pi \sin x + A_+ \sin \pi + B_+ \cos \pi \Rightarrow B_+ = -\sin x$

$\therefore \tilde{G}(s, x) = \frac{2 \cos s \sin x}{\pi} + \begin{cases} A_- \sin s & 0 < s < x \\ A_+ \sin s - \sin x \cos s & x < s < \pi \end{cases}$

CONTINUITY  $\tilde{G}(x_-, x) = x \cos x \sin x + A_- \sin x = \tilde{G}(x_+, x) = A_+ \sin x - \sin x \cos x + x \cos x \sin x$

$\therefore (A_+ - A_-) \sin x = \sin x \cos x$

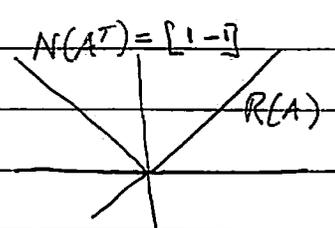
JUMP:  $\int_{x-E}^{x+E} G_{ss} + G ds = \int_{x-E}^{x+E} \delta(s-x) - 2 \sin s \sin x ds = 1$

$G_s(x_+, x) - G_s(x_-, x) = A_+ \cos x + \sin^2 x - A_- \cos x = 1$

$(A_+ - A_-) \cos x = 1 - \sin^2 x$

$A_+ - A_- = \frac{1}{\cos x} - \frac{\sin x \cdot \sin x}{\cos x}$

$$\begin{bmatrix} 1 & -1 \\ A_+ & A_- \end{bmatrix} = \begin{bmatrix} -\cos x \\ \frac{1}{\cos x} - \tan x \sin x \end{bmatrix} = b$$



WHICH IS AN UNDER DETERMINED SYSTEM. FOR A SOLUTION TO EXIST

WE MUST HAVE THAT  $\sqrt{1} b = [1 -1] \begin{bmatrix} -\cos x \\ \frac{1}{\cos x} - \tan x \sin x \end{bmatrix} = \frac{-\cos^2 x - 1 + \sin^2 x}{\cos x} = 0$

$$A_- = A_+ - \cos x$$

$$\tilde{G}(s, x) = s \cos s \sin x + A_+ \sin s + \begin{cases} -s \sin s \cos x & 0 < s < x \\ -s \sin x \cos s & x < s < \pi \end{cases}$$

$$\therefore u(x) = \int_0^{\pi} \tilde{G}(s, x) f(s) ds - \frac{2}{\pi} \int_0^{\pi} u(s) \sin s \sin x ds$$

$$= \int_0^{\pi} [s \cos s] f(s) ds \cdot \sin x + A_+ \int_0^{\pi} f(s) \sin s ds - \frac{2}{\pi} \int_0^{\pi} u(s) \sin s ds \sin x$$

*SOLVABILITY*

$$- \left\{ \int_0^x \sin s f(s) ds \cos x + \int_x^{\pi} \cos s f(s) ds \sin x \right\}$$

$$= C \sin x - \left( \int_0^x \sin s f(s) ds \cos x + \int_x^{\pi} \cos s f(s) ds \right) \sin x$$

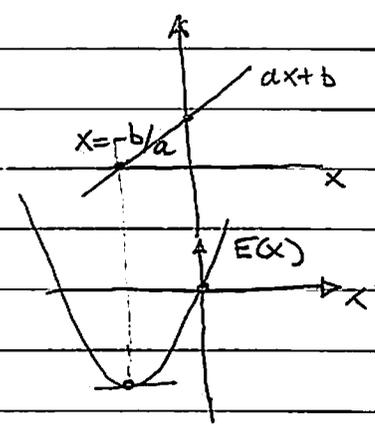
LECTURE 18 VARIATIONAL METHODS FOR BOUNDARY VALUE PROBLEMS

• OFTEN IT IS MORE CONVENIENT TO POSE A PROBLEM AS A MINIMIZATION PROBLEM

EG1: CONSIDER SOLVING  $ax+b=0$

MINIMIZE  $E(x) = \frac{1}{2}ax^2 + bx$

$0 = E'(x) = ax + b$  A NECESSARY CONDITION



EG2: LINEAR ALGEBRA

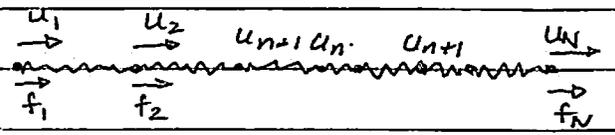
CONSIDER SOLVING  $Ax = b$  WHERE  $A$  IS SYMMETRIC

$E(x) = \frac{1}{2}x^T Ax - x^T b$

$0 = \frac{\partial E}{\partial x} = Ax - b$

EG3: EQUILIBRIUM OF A MASS-SPRING SYSTEM

$P.E = E(u) = \sum_{n=2}^N \frac{1}{2} k(u_n - u_{n-1})^2 + \sum_{n=1}^N f_n u_n$



$0 = \frac{\partial E}{\partial u_m} = k(u_m - u_{m-1}) + k(u_{m+1} - u_m)(-1) + f_m$

$\therefore -k(u_{m+1} - 2u_m + u_{m-1}) = -f_m$

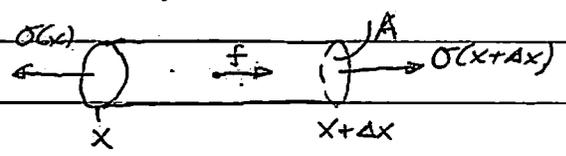
CONTINUUM LIMIT

$k \Delta x^2 u'' \sim f$

$\left(\frac{k \Delta x}{A}\right) u'' \sim \frac{f}{A \Delta x}$

$E u'' = f$

EG4: EQUILIBRIUM OF AN ELASTIC BAR



$(\sigma(x+\Delta x) - \sigma(x))A = f \cdot A \Delta x$

$\frac{\partial \sigma}{\partial x} = f$

HOOKE'S LAW  $\sigma = E \frac{\partial u}{\partial x}$

$(E u_x)_x = f$