

LECTURE 17. THE FREDHOLM ALTERNATIVE AND THE MODIFIED GREEN'S FUNCTION

RECALL  $Lu = -(pu')' + qu = f(x)$

$B_0 u = 0 = B_1 u$

THEN BY EIGENFUNCTION EXPANSION

$$u(x) = \sum_{n=1}^{\infty} \frac{\phi_n(x) \int_0^1 \phi_n(s) f(s) ds}{\lambda_n}$$

WHERE  $L\phi_n = \lambda_n \tau(x) \phi_n(x)$   $\lambda_n$  &  $\phi_n$  ARE EIGENVALUES & EIGENFUNCTIONS

FREDHOLM ALTERNATIVE:

CONSIDER THE ESSENTIALLY SELF-ADJOINT BVP

$$Lu = -(pu')' + qu = f(x) \quad B_0 u = 0 = B_1 u \quad (SA)$$

THEN EITHER

(I)  $Lu = f$  HAS A UNIQUE SOLUTION

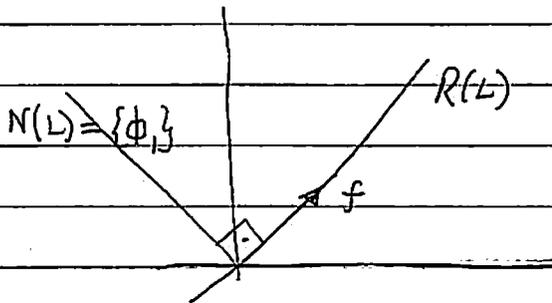
OR (II) THE EIGENVALUE PROBLEM  $Lu = \lambda \tau u + BC$  HAS A ZERO EIGENVALUE

$\lambda_1 = 0$  WITH EIGENFUNCTION  $\phi_1(x) \neq 0$ .

IN THIS CASE (SA) HAS A SOLUTION IF AND ONLY IF

$$\int_0^1 f(x) \phi_1(x) dx = 0 \quad (\text{SOLVABILITY CONDITION})$$

AND  $u(x)$  IS ONLY DETERMINED UP TO AN ARBITRARY MULTIPLE OF  $\phi_1(x)$ .



THE MODIFIED GREEN'S FUNCTION

IF THERE IS A  $\tilde{u} \neq 0$  SUCH THAT  $L\tilde{u} = 0 + B_0\tilde{u} = 0 = B_1\tilde{u}$   
THEN A SOLUTION TO  $Lu = f + BC$  ONLY EXISTS PROVIDED  $f$   
SATISFIES THE SOLVABILITY CONDITION

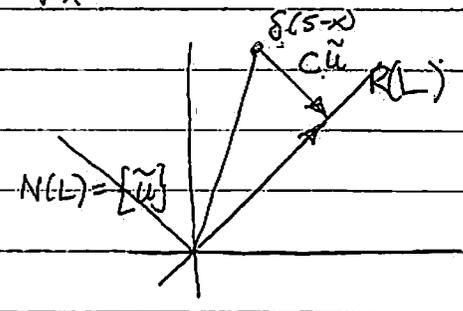
$$\int_0^1 \tilde{u}(x) f(x) dx = 0$$

• THUS A GREEN'S FUNCTION DOES NOT EXIST UNLESS

$$0 = \int_0^1 \tilde{u}(s) \delta(s-x) ds = \tilde{u}(x) \quad \forall x$$

• WHAT ABOUT LOOKING FOR A MODIFIED  
GREEN'S FUNCTION  $\tilde{G}$  OF THE FORM

$$L_s \tilde{G}(s, x) = \delta(s-x) + C\tilde{u}$$
  
$$B_0 \tilde{G} = 0 = B_1 \tilde{G}$$



WHERE  $C$  IS CHOSEN SO THAT

$$0 = \int_0^1 \tilde{u}(s) \{ \delta(s-x) + C\tilde{u}(s) \} ds \Rightarrow C = -\tilde{u}(x) / \int_0^1 \tilde{u}(s)^2 ds$$

$$L_s \tilde{G}(s, x) = \delta(s-x) - \tilde{u}(x) / (\tilde{u}, \tilde{u})$$

THEN 
$$u(x) = \tilde{u}(x) + \int_0^1 \tilde{G}(s, x) f(s) ds$$

EXAMPLES  $Lu = u'' = f \quad (u'(0) = 0 = u'(1))$

$\tilde{u} = 1$  IS A NONTRIVIAL SOLUTION TO  $L\tilde{u} = 0\tilde{u} + BC$

THE MODIFIED GREEN'S FUNCTION  $\tilde{G}$  IS GIVEN BY A SOLUTION TO

$$L_s \tilde{G}(s, x) = \delta(s-x) + C\tilde{u} = \delta(s-x) - 1 \quad C = -1/(L\tilde{u}, \tilde{u})$$

$$\tilde{G}_s(0, x) = 0 = \tilde{G}_s(1, x)$$

PARTICULAR SOLUTION:  $v_{ss} = -1 \quad v_s = -s + A \quad v = -\frac{s^2}{2} + As + B$

LET 
$$\tilde{G}(s, x) = -\frac{s^2}{2} + \begin{cases} A - s + B_- & \tilde{G}_s = -s + \begin{cases} A_- \\ A_+ \end{cases} \\ A_+ + s + B_+ & \end{cases}$$

$$\tilde{G}_s(0, x) = A_- = 0 \quad \tilde{G}_s(1) = -1 + A_+ = 0 \Rightarrow A_+ = 1$$

$$\tilde{G}(s, x) = -\frac{s^2}{2} + \begin{cases} B_- & 0 < s < x \\ s + B_+ & x < s < 1 \end{cases}$$

CONTINUITY  $\tilde{G}(x_-, x) = -\frac{x^2}{2} + B_- = \tilde{G}(x_+, x) = -\frac{x^2}{2} + x + B_+ \quad B_- = x + B_+$

JUMP:  $L_s \tilde{G} = \tilde{G}_{ss} = \delta(s-x) - 1$   

$$\int_{x-\epsilon}^{x+\epsilon} \tilde{G}_{ss} ds = \tilde{G}_s(x_+, x) - \tilde{G}_s(x_-, x) = \int_{x-\epsilon}^{x+\epsilon} [\delta(s-x) - 1] ds = 1$$
  
 $\therefore \{-x + 1\} - \{-x + 0\} = 1$  SATISFIED AUTOMATICALLY

$$\therefore \tilde{G}(s, x) = -\frac{s^2}{2} + \begin{cases} x + B_+ \\ s + B_+ \end{cases}$$

$$( \tilde{G}, Lu ) = ( u, L \tilde{G} ) = u(x) - \int_0^1 u(s) 1 ds$$

$$u(x) = \int_0^1 u(s) ds + \int_0^1 \left[ -\frac{s^2}{2} + B_+ + \begin{cases} x \\ s \end{cases} \right] f(s) ds$$

$$= C 1 + x \int_0^x f(s) ds + \int_x^1 s f(s) ds$$