LECTURE 14: GREEN'S FUNCTIONS FOR VARIABLE COEFFICIENT 2ND ORDER BOUNDARY VALUE PROBLEMS

CONSIDER THE VARIABLE COEFFICIENT SECOND ORDER 2POINT BOUNDARY VALUE PROBLEM

$$Lu = a_0(x)u'' + a_1(x)u' + a_2 u = f(x) \quad \text{LET} \quad (u,v) = \int_{x_0}^{x_1} uv\,dx$$

$$u(x_0) = u_0 \qquad u(x_1) = u_1$$

$$(v, Lu) = \int_{x_0}^{x_1} v\{a_0 u'' + a_1 u' + a_2 u\}\,dx$$

$$= [v a_0 u' + v a_1 u]_{x_0}^{x_1} - \int_{x_0}^{x_1} u'(a_0 v)' + u(a_1 v)' - u a_2 v\,dx$$

$$= [v a_0 u' + v a_1 u - u(a_0 v)']_{x_0}^{x_1} + \int_{x_0}^{x_1} u\underbrace{\{(a_0 v)'' - (a_1 v) + a_2 v\}}_{L^* v}\,dx$$

$$= [v a_0 u' + v a_1 u - u a_0' v - u a_0 v']_{x_0}^{x_1} + \int_{x_0}^{x_1} u L^* v\,dx$$

IF WE CHOOSE $v(x_0) = 0 = v(x_1)$ AND $L_s^* v = \delta(s-x)$

THEN $$\boxed{u(x) = \int_{x_0}^{x_1} \bar{v}(s,x) f(s)\,ds + u_1 a_0(x_1)\bar{v}_s(x_1,x) - u_0 a_0(x_0)\bar{v}_s(x_0,x)}$$

EG: $Lu = x^2 u'' - 2xu' - 4u = f(x) \qquad 0<x<1 \qquad u(0)<\infty \quad u(1)=1$

$$(v, Lu) = \int_0^1 v\{x^2 u'' - 2xu' - 4u\}\,dx$$

$$= [v x^2 u' - 2xvu - (x^2 v)' u]_0^1 + \int_0^1 u\{(x^2 v)'' + (2xv)' - 4v\}\,dx$$

$$= [v x^2 u' - 2xvu - 2xvu - x^2 v' u]_0^1 + \int_0^1 u L^* v\,dx$$

CHOOSE $v(0) < \infty$ AND $v(1) = 0$ AND $L_s^* \bar{v}(s,x) = (s^2 \bar{v})'' + (2s\bar{v})' - 4\bar{v} = \delta(s-x)$

THEN $u(x) = \int_0^1 \bar{v}(s,x) f(s)\,ds + \bar{v}_s^*(1,x) \cdot 1$

$$L_s^* v = (s^2 v'' + 4sv' + 2v) + (2sv' + 2v) - 4v = s^2 v'' + 6sv'$$

HOMOG EQ: C-E $v = s^\gamma \Rightarrow \gamma(\gamma-1) + 6\gamma = \gamma^2 + 5\gamma = 0 \Rightarrow \gamma = 0$ OR $\gamma = -5$

$$\bar{v}(s,x) = \begin{cases} A_- + B_- s^{-5} & 0<s<x \\ A_+ + B_+ s^{-5} & x<s<1 \end{cases}$$

BC: $\bar{v} < \infty$ AS $s \to 0 \Rightarrow B_- = 0 \quad \bar{v}(1,x) = A_+ + B_+ = 0 \Rightarrow B_+ = -A_+$

$$\therefore \quad \bar{v}(s,x) = \begin{cases} A_- & 0<s<x \\ A_+(1-s^{-5}) & x<s<1 \end{cases} \qquad v_s = \begin{cases} 0 & 0<s<x \\ 5A_+ s^{-6} & x<s<1 \end{cases}$$

CONTINUITY: $\bar{v}(x_-,x) = A_- = A_+(1-x^{-5}) = \bar{v}(x_+,x)$

JUMP $$\int_{x-\varepsilon}^{x+\varepsilon} (s^2\bar{v})'' + (2s\bar{v})' - 4\bar{v}\,ds = \int_{x-\varepsilon}^{x+\varepsilon} \delta(s-x)\,ds = 1$$

$$[(s^2 v)' + 2sv - 4v]_{x-}^{x+} = [s^2 v' + 4sv - 4v]_{x-}^{x+} = x^2 [\![v']\!] = 1$$

$$\therefore \quad x^2\{5A_+ x^{-6} - 0\} = 1 \qquad A_+ = x^4/5$$

$$\therefore \quad A_- = \tfrac{1}{5}(x^4 - x^{-1})$$

$$\therefore \quad V(s,x) = \begin{cases} \frac{1}{5}(x^4 - x^{-1}) & 0<s<x \\ \frac{x^4}{5}(1-s^{-5}) & x<s<1 \end{cases} \qquad V_s = \begin{cases} 0 & 0<s<x \\ x^4 s^{-6} & x<s<1 \end{cases}$$

$$\therefore \quad u(x) = x^4 + \int_0^1 \frac{1}{5}\begin{Bmatrix} x^4 - x^{-1} \\ x^4(1-s^{-5}) \end{Bmatrix} f(s)\,ds.$$

<u>SELF-ADJOINT OPERATORS</u>

IF $L^* = L$ THEN L IS SAID TO BE FORMALLY SELF-ADJOINT.

UNDER WHAT CONDITIONS IS $L^* = L$?

$$Lu = a_0 u'' + a_1 u' + a_2 u$$

$$L^* u = (a_0 u)'' - (a_1 u)' + a_2 u$$

$$= (a_0 u' + a_0' u)' - (a_1 u' + a_1' u) + a_2 u$$

$$= (a_0 u'' + 2a_0' u' + a_0'' u) - (a_1 u' + a_1' u) + a_2 u$$

$$= a_0 u'' + (2a_0' - a_1) u' + (a_0'' - a_1' + a_2) u$$

u'] $\quad 2a_0' - a_1 = a_1 \Rightarrow \boxed{a_0' = a_1}$

u] $\quad \boxed{a_0'' = a_1'}$ SAME

THUS L IS FORMALLY SELF ADJOINT IF $a_0' = a_1$

THUS $Lu = a_0 u'' + a_1 u' + a_2 u = a_0 u'' + a_0' u' + a_2 u = (a_0 u')' + a_2 u$.

CAN WE MULTIPLY L BY SOME FACTOR $F(x)$ TO MAKE $\mathcal{L} = F(x) L$ FORMALLY SELF ADJOINT?

TO BE FSA $\quad (F a_0)' = F a_1$

$$\Rightarrow \quad F' a_0 + a_0' F - a_1 F = 0$$

$$F' + \left(\frac{a_0'}{a_0} - \frac{a_1}{a_0}\right) F = 0$$

$$\left[e^{\int \left(\frac{a_0'}{a_0} - \frac{a_1}{a_0}\right) dx} F \right]' = 0$$

$$\therefore \quad \boxed{F = \frac{e^{\int \frac{a_1}{a_0}}}{a_0}} \qquad \text{ABEL'S FORMULA.}$$

eg: $\quad Lu = x^2 u'' - 2x u' - 4u$

$$F = \frac{e^{\int \frac{-2x}{x^2} dx}}{x^2} = \frac{e^{-2\ln x}}{x^2} = x^{-4}$$

$$x^{-2} u'' - 2x^{-3} u' - 4x^{-4} u = (x^{-2} u')' - 4x^{-4} u = 0$$