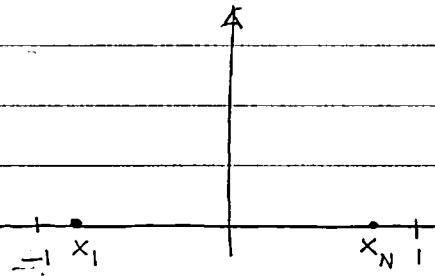


LECTURE 10 GAUSS QUADRATURE (CONTINUED)

LAST TIME

GAUSS-LEGENDRE QUADRATURE

$$\int_{-1}^1 f(x) dx = \int_{-1}^1 \phi_{2N-1}(x) dx + E_{2N-1}$$



$$= \sum_{k=1}^{2N} f_k \underbrace{\int_{-1}^1 \ell^{(2N-1)}(x) dx}_{w_k} + E_{2N-1}$$

$$\text{FOR } k \geq N+1 \quad w_k = \int_{-1}^1 (x-x_1)\dots(x-x_N)(x-x_{N+1})\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_{2N}) dx$$

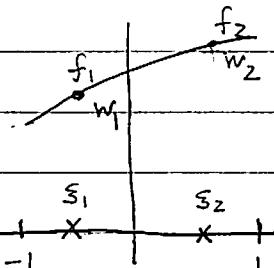
$\underbrace{(x_k-x_1)\dots(x_k-x_N)(x_k-x_{N+1})\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_{2N})}$

IF WE CHOOSE x_1, \dots, x_N TO BE THE ZEROS OF $P_N(x)$ THE LEGENDRE POLYNOMIAL OF DEG N

$$\boxed{\int_{-1}^1 f(x) dx = \sum_{k=1}^N f_k w_k + E_{2N-1}}$$

EXAMPLE: $N=2$ SHOULD INTEGRATE A POLYNOMIAL OF DEG 2N-1 EXACTLY

$$\int_{-1}^1 a_0 + a_1 x + a_2 x^2 + a_3 x^3 dx = 2a_0 + \frac{2}{3}a_2$$



$$w_1 \{ a_0 + a_1 \xi_1 + a_2 \xi_1^2 + a_3 \xi_1^3 \} + w_2 \{ a_0 + a_1 \xi_2 + a_2 \xi_2^2 + a_3 \xi_2^3 \}$$

$$w_1 = w_2 \\ \xi_2 = -\xi_1 \\ = w_1 \{ 2a_0 + a_1 (\xi_1 + \xi_1) + a_2 (\xi_1^2 + \xi_1^2) + a_3 (\xi_1^3 + \xi_1^3) \}$$

ASSUME $w_1 = w_2$

$$\xi_1 = -\xi_2$$

$$a_0] 2w_1 = 2 \Rightarrow w_1 = 1 = w_2$$

$$a_2] 2\xi_1^2 = \frac{2}{3} \Rightarrow \xi_1 = -\frac{1}{\sqrt{3}} = -\xi_2$$

$$\therefore \int_{-1}^1 f(x) dx = [f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})] + E_3 \quad \text{RECALL } P_2(x) = \frac{1}{2}(3x^2 - 1) = 0 \quad x = \pm \frac{1}{\sqrt{3}}$$

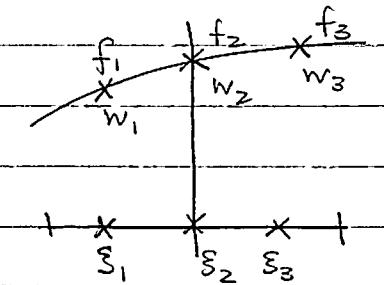
2

N=3: SHOULD INTEGRATE A POLYNOMIAL OF DEGREE 5 EXACTLY

$$\int_1^5 a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 dx$$

$$= 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4$$

$$= w_1 f(\xi_1) + w_2 f(\xi_2) + w_3 f(\xi_3)$$



$\xi_2 = 0$ BY SYMMETRY

$$w_1 \{a_0 - a_1 \xi_3 + a_2 \xi_2^2 - a_3 \xi_3^3 + a_4 \xi_3^4 - a_5 \xi_3^5\}$$

$$+ w_2 \{a_0\}$$

$$+ w_1 \{a_0 + a_1 \xi_3 + a_2 \xi_2^2 + a_3 \xi_3^3 + a_4 \xi_2^4 + a_5 \xi_3^5\}$$

$$a_0] \quad 2 = 2w_1 + w_2$$

$$a_2] \quad \frac{2}{3} = 2w_1 \xi_3^2$$

$$a_4] \quad \frac{2}{5} = 2w_1 \xi_3^4 \Rightarrow (2w_1 \xi_2^2) \xi_3^2 = \frac{2}{3} \xi_3^2 \Rightarrow \xi_3 = \sqrt{\frac{3}{5}} = -\xi_1$$

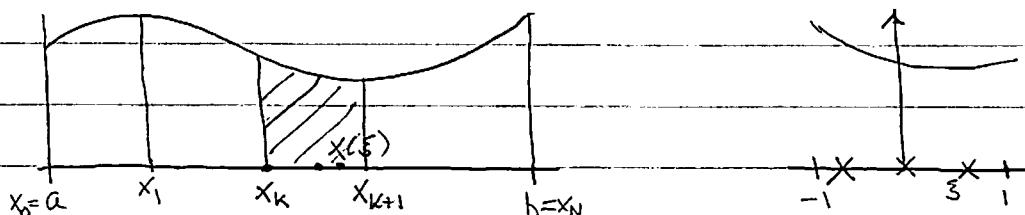
$$2w_1 \xi_3^2 = 2w_1 \left(\frac{3}{5}\right) = \frac{2}{3} \Rightarrow w_1 = \frac{5}{9} \quad w_2 = 2 - 2w_1 = 2 - \frac{10}{9} = \frac{8}{9}$$

$$\int_{-1}^5 f(x) dx = \frac{5}{9} f(-\sqrt{\frac{3}{5}}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{\frac{3}{5}})$$

$$RECALL \quad P_n = \frac{(2n-1)}{n} \times P_n - \frac{(n-1)}{n} P_{n-2}$$

$$P_3 = \frac{5}{3} \times \frac{1}{2} (3x^2 - 1) - \frac{2}{3} \times x = \frac{1}{3} (5x^2 - 3)x \quad P_3(\pm \sqrt{\frac{3}{5}}) = 0$$

IMPLEMENTATION IN A COMPOSITE RULE



$$x(\xi) = \frac{(x_{k+1} + x_k)}{2} + \frac{(x_{k+1} - x_k)}{2} \xi \quad dx = h_k ds$$

$$\begin{aligned} \int_a^b f(x) dx &= \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} f(x) dx = \sum_{k=0}^{N-1} h_k \int_{-1}^1 f(\bar{x}_k + h_k s) ds \\ &= \sum_{k=0}^{N-1} h_k \left[\frac{5}{9} f(\bar{x}_k - h_k \sqrt{\frac{3}{5}}) + \frac{8}{9} f(\bar{x}_k) + \frac{5}{9} f(\bar{x}_k + h_k \sqrt{\frac{3}{5}}) \right] \end{aligned}$$

ADAPTIVE GAUSS INTEGRATION

3
1

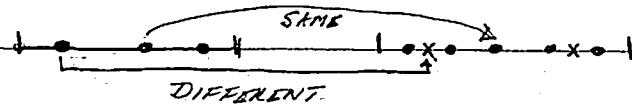
- WE HAVE SEEN ADAPTIVE TRAPEZOIDAL INTEGRATION



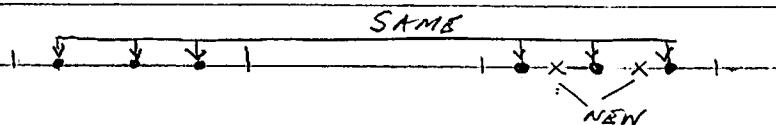
+ EXTRAPOLATION \rightarrow ROMBERG.

\rightarrow SUCCESSIVE APPROXIMATIONS USE PREVIOUS VALUES

- PROBLEM WITH GAUSS INTEGRATION IS THAT SUCCESSIVE APPROXIMATIONS



- GAUSS-PATTERSON & GAUSS-KRONROD INTEGRATION (quadgk)



G7+K15 ERROR ESTIMATE FROM |G7-K15|